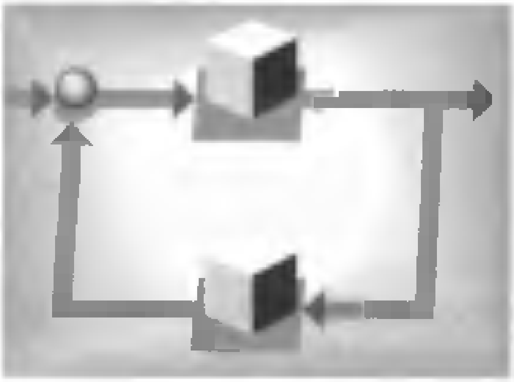
T W E L V E



Design via State Space

State Space | This chapter covers only state-space methods.

Chapter Objectives

In this chapter you will learn the following:

- How to design a state-feedback controller using pole placement to meet transient response specifications
- How to design an observer for systems where the states are not available to the controller
- How to design steady-state error characteristics for systems represented in state space

Case Study Objectives

You will be able to demonstrate your knowledge of the chapter objectives with case studies as follows:

- Given the antenna azimuth position control system shown on the front endpapers, you will be able to specify all closed-loop poles and then design a state-feedback controller to meet transient response specifications.
- Given the antenna azimuth position control system shown on the front endpapers, you will be able to design an observer to estimate the states.
- Given the antenna azimuth position control system shown on the front endpapers, you will be able to combine the controller and observer designs into a viable compensator for the system.

Figure 12.1
An automatic pharmacy system showing a robot picking up drugs to deposit in boxes for individual patients at a hospital

12.1 Introduction

Chapter 3 introduced the concepts of state-space analysis and system modeling. We showed that state-space methods, like transform methods, are simply tools for analyzing and designing feedback control systems. However, state-space techniques can be applied to a wider class of systems than transform methods. Systems with nonlinearities, such as that shown in Figure 12.1, and multiple-input, multiple-output systems are just two of the candidates for the state-space approach. In this book, however, we apply the approach only to linear systems.

In Chapters 9 and 11 we applied frequency domain methods to system design. The basic design technique is to create a compensator in cascade with the plant or in the feedback path that has the correct additional poles and zeros to yield a desired transient response and steady-state error.

One of the drawbacks of frequency domain methods of design, using either root locus or frequency response techniques, is that after designing the location of the dominant second-order pair of poles, we keep our fingers crossed, hoping that the higher-order poles do not affect the second-order approximation. What we would like to be able to do is specify *all* closed-loop poles of the higher-order system. Frequency domain methods of design do not allow us to specify all poles in systems of order higher than two because they do not allow for a sufficient number of unknown parameters to place all of the closed-loop poles uniquely. One gain to adjust, or compensator pole and zero to select, does not yield a sufficient number of parameters to place all the closed-loop poles at desired locations. Remember, to place n unknown quantities, you need n adjustable parameters. State-space methods solve this problem by introducing into the system (1) other adjustable parameters and

(2) the technique for finding these parameter values, so that we can properly place all poles of the closed-loop system.[1]

On the other hand, state-space methods do not allow the specification of closed-loop zero locations, which frequency domain methods do allow through placement of the lead compensator zero. This is a disadvantage of state-space methods, since the location of the zero does affect the transient response. Also, a state-space design may prove to be very sensitive to parameter changes.

Finally, there is a wide range of computational support for state-space methods; many software packages support the matrix algebra required by the design process. However, as mentioned before, the advantages of computer support are balanced by the loss of graphic insight into a design problem that the frequency domain methods yield.

This chapter should be considered only an introduction to state-space design; we introduce one state-space design technique and apply it only to linear systems. Advanced study is required to apply state-space techniques to the design of systems beyond the scope of this textbook.

12.2 Controller Design

This section shows how to introduce additional parameters into a system so that we can control the location of all closed-loop poles. An nth-order feedback control system has an nth-order closed-loop characteristic equation of the form

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{12.1}$$

Since the coefficient of the highest power of s is unity, there are n coefficients whose values determine the system's closed-loop pole locations. Thus, if we can introduce n adjustable parameters into the system and relate them to the coefficients in Eq. (12.1), all of the poles of the closed-loop system can be set to any desired location.

Topology for Pole Placement

In order to lay the groundwork for the approach, consider a plant represented in state space by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{12.2a}$$

$$y = \mathbf{C}\mathbf{x} \tag{12.2b}$$

and shown pictorially in Figure 12.2(a), where light lines are scalars and the heavy lines are vectors.

In a typical feedback control system, the output, y, is fed back to the summing junction. It is now that the topology of the design changes. Instead of feeding back

[1]This is an advantage as long as we know where to place the higher-order poles, which is not always the case. One course of action is to place the higher-order poles far from the dominant second-order poles or near a closed-loop zero to keep the second-order system design valid. Another approach is to use optimal control concepts, which are beyond the scope of this text.

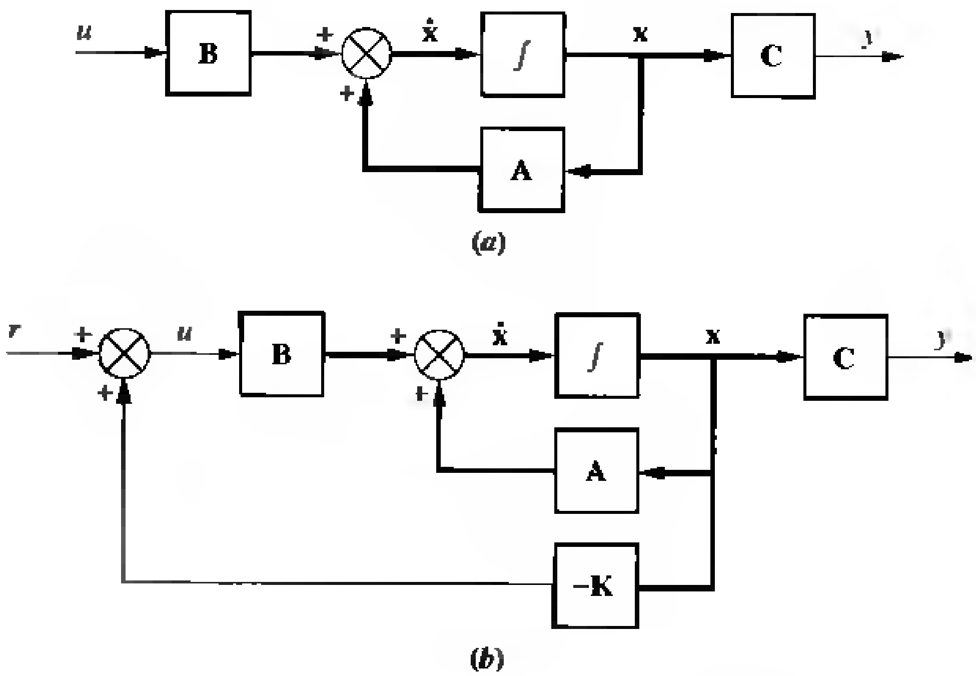


Figure 12.2
a. State-space representation of a plant;
b. plant with state-feedback

y, what if we feed back all of the state variables? If each state variable is fed back to the control, u, through a gain, k_i, there would be n gains, k_i, that could be adjusted to yield the required closed-loop pole values. The feedback through the gains, k_i, is represented in Figure 12.2(*b*) by the feedback vector $-\mathbf{K}$.

The state equations for the closed-loop system of Figure 12.2(*b*) can be written by inspection as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x} + r) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}r \tag{12.3a}$$

$$y = \mathbf{C}\mathbf{x} \tag{12.3b}$$

Before continuing, you should have a good idea of how the feedback system of Figure 12.2(*b*) is actually implemented. As an example, assume a plant signal-flow graph in phase-variable form, as shown in Figure 12.3(*a*). Each state variable is then fed back to the plant's input, u, through a gain, k_i, as shown in Figure 12.3(*b*). Although we will cover other representations later in the chapter, the phase-variable form, with its typical lower companion system matrix, or the controller canonical form, with its typical upper companion system matrix, yields the simplest evaluation of the feedback gains. In the ensuing discussion, we use the phase-variable form to develop and demonstrate the concepts. End-of-chapter problems will give you an opportunity to develop and test the concepts for the controller canonical form.

The design of state-variable feedback for closed-loop pole placement consists of equating the characteristic equation of a closed-loop system, such as that shown in Figure 12.3(*b*), to a desired characteristic equation and then finding the values of the feedback gains, k_i.

If a plant like that shown in Figure 12.3(*a*) is of high order and not represented in phase-variable or controller canonical form, the solution for the k_i's can be intricate. Thus, it is advisable to transform the system to either of these forms, design the k_i's, and then transform the system back to its original representation. We perform

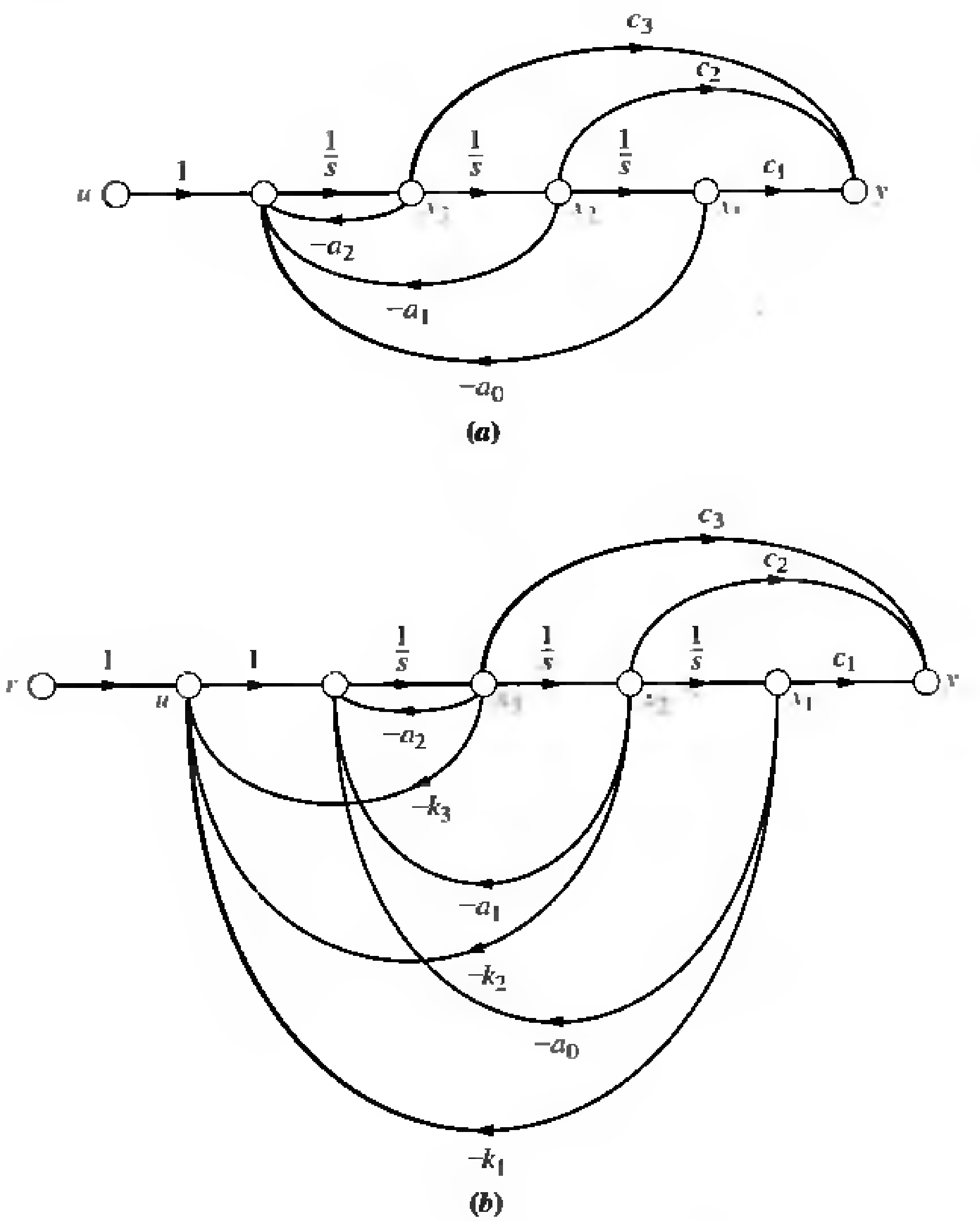


Figure 12.3
a. Phase-variable representation for plant;
b. plant with state-variable feedback

this conversion in Section 12.4, where we develop a method for performing the transformations. Until then, let us direct our attention to plants represented in phase-variable form.

Pole Placement for Plants in Phase-Variable Form

To apply pole-placement methodology to plants represented in phase-variable form. we take the following steps:

1. Represent the plant in phase-variable form.
2. Feed back each phase variable to the input of the plant through a gain, k_i.
3. Find the characteristic equation for the closed-loop system represented in step 2.

4. Decide upon all closed-loop pole locations and determine an equivalent characteristic equation.
5. Equate like coefficients of the characteristic equations from steps 3 and 4 and solve for k_i.

Following these steps, the phase-variable representation of the plant is given by Eq. (12.2), with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$\mathbf{C} = [c_1 \quad c_2 \quad \cdots \quad c_n] \tag{12.4}$$

The characteristic equation of the plant is thus

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{12.5}$$

Now form the closed-loop system by feeding back each state variable to u, forming

$$u = -\mathbf{Kx} \tag{12.6}$$

where

$$\mathbf{K} = [k_1 \quad k_2 \quad \cdots \quad k_n] \tag{12.7}$$

The k_i's are the phase variables' feedback gains.

Using Eq. (12.3a) with Eqs. (12.4) and (12.7), the system matrix, $\mathbf{A} - \mathbf{BK}$, for the closed-loop system is

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \cdots & -(a_{n-1} + k_n) \end{bmatrix} \tag{12.8}$$

Since Eq. (12.8) is in phase-variable form, the characteristic equation of the closed-loop system can be written by inspection as

$$\det(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = s^n + (a_{n-1} + k_n)s^{n-1} + (a_{n-2} + k_{n-1})s^{n-2} \\ + \cdots (a_1 + k_2)s + (a_0 + k_1) = 0 \tag{12.9}$$

Notice the relationship between Eqs. (12.5) and (12.9). For plants represented in phase-variable form, we can write by inspection the closed-loop characteristic equation from the open-loop characteristic equation by adding the appropriate k_i to each coefficient.

Now assume that the desired characteristic equation for proper pole placement is

$$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_2s^2 + d_1s + d_0 = 0 \qquad (12.10)$$

where the d_i's are the desired coefficients. Equating Eqs. (12.9) and (12.10), we obtain

$$d_i = a_i + k_{i+1} \qquad i = 0, 1, 2, \ldots, n-1 \qquad (12.11)$$

from which

$$k_{i+1} = d_i - a_i \qquad (12.12)$$

Now that we have found the denominator of the closed-loop transfer function, let us find the numerator. For systems represented in phase-variable form, we learned that the numerator polynomial is formed from the coefficients of the output coupling matrix, **C**. Since Figures 12.3(*a*) and (*b*) are both in phase-variable form and have the same output coupling matrix, we conclude that the numerators of their transfer functions are the same. Let us look at a design example.

Example 12.1

Controller design for phase-variable form

Problem Given the plant

$$G(s) = \frac{20(s+5)}{s(s+1)(s+4)} \qquad (12.13)$$

design the phase-variable feedback gains to yield 9.5% overshoot and a settling time of 0.74 second.

Solution We begin by calculating the desired closed-loop characteristic equation. Using the transient response requirements, the closed-loop poles are $-5.4 \pm j7.2$. Since the system is third-order, we must select another closed-loop pole. The closed-loop system will have a zero at -5, the same as the open-loop system. We could select the third closed-loop pole to cancel the closed-loop zero. However, to demonstrate the effect of the third pole and the design process, including the need for simulation, let us choose -5.1 as the location of the third closed-loop pole.

Now draw the signal-flow diagram for the plant. The result is shown in Figure 12.4(*a*). Next feed back all state variables to the control, u, through gains k_i, as shown in Figure 12.4(*b*).

Writing the closed-loop system's state equations from Figure 12.4(*b*), we have

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(4+k_2) & -(5+k_3) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (12.14a)$$

$$y = \begin{bmatrix} 100 & 20 & 0 \end{bmatrix}\mathbf{x} \qquad (12.14b)$$

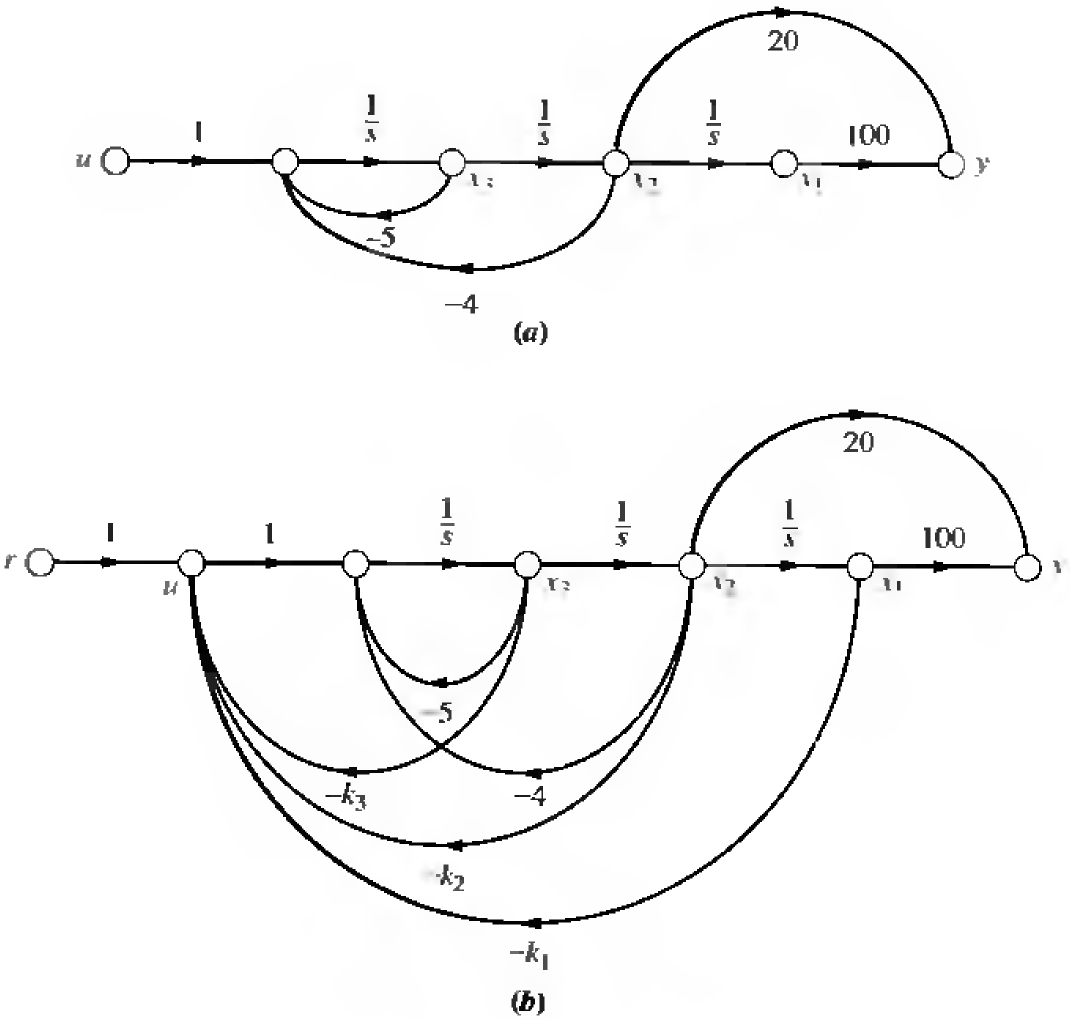


Figure 12.4
a. Phase-variable representation for plant of Example 12.1; **b.** plant with state-variable feedback

Comparing Eq. (12.14) to Eq. (12.3), we identify the closed-loop system matrix as

$$\mathbf{A}-\mathbf{BK}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -k_1 & -(4+k_2) & -(5+k_3)\end{bmatrix} \tag{12.15}$$

To find the closed-loop system's characteristic equation, form

$$\det(s\mathbf{I}-(\mathbf{A}-\mathbf{BK}))=s^3+(5+k_3)s^2+(4+k_2)s+k_1=0 \tag{12.16}$$

This equation must match the desired characteristic equation,

$$s^3+15.9s^2+136.08s+413.1=0 \tag{12.17}$$

formed from the poles $-5.4+j7.2$, $-5.4-j7.2$, and -5.1, which were previously determined.

Equating the coefficients of Eqs. (12.16) and (12.17), we obtain

$$k_1=413.1;\quad k_2=132.08;\quad k_3=10.9 \tag{12.18}$$

Finally, the zero term of the closed-loop transfer function is the same as the zero term of the open-loop system, or $(s+5)$.

Using Eq. (12.14), we obtain the following state-space representation of the closed-loop system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -413.1 & -136.08 & -15.9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \tag{12.19a}$$

$$y = [100 \quad 20 \quad 0]\mathbf{x} \tag{12.19b}$$

The transfer function is

$$T(s) = \frac{20(s+5)}{s^3 + 15.9s^2 + 136.08s + 413.1} \tag{12.20}$$

Figure 12.5, a simulation of the closed-loop system, shows 11.5% overshoot and a settling time of 0.8 second. A redesign with the third pole canceling the zero at −5 will yield performance equal to the requirements.

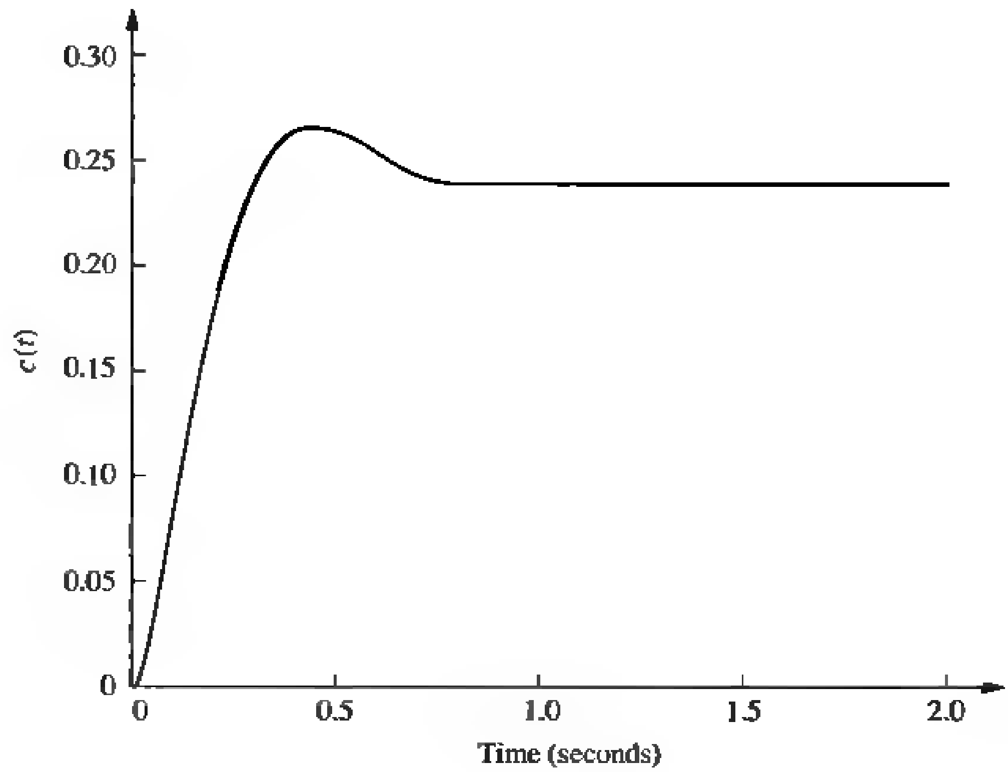


Figure 12.5 Simulation of closed-loop system of Example 12.1

Since the steady-state response approaches 0.24 instead of unity, there is a large steady-state error. Design techniques to reduce this error are discussed in Section 12.8.

MATLAB

Students who are using MATLAB should now run ch12p1 in Appendix B. You will learn how to use MATLAB to design a controller for phase variables using pole placement. MATLAB will plot the step response of the designed system. This exercise solves Example 12.1 using MATLAB.

Skill-Assessment Exercise 12.1

Problem For the plant

$$G(s) = \frac{100(s+10)}{s(s+3)(s+12)}$$

represented in the state space in phase-variable form by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -36 & -15 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{x} = \begin{bmatrix} 1000 & 100 & 0 \end{bmatrix}\mathbf{x}$$

design the phase-variable feedback gains to yield 5% overshoot and a peak time of 0.3 second.

Answer $\mathbf{K} = [2094 \quad 373.1 \quad 14.97]$

The complete solution is on the accompanying CD-ROM.

In this section we showed how to design feedback gains for plants represented in phase-variable form in order to place all of the closed-loop system's poles at desired locations on the s-plane. On the surface it appears that the method should always work for any system. However, this is not the case. The conditions that must exist in order to uniquely place the closed-loop poles where we want them is the topic of the next section.

12.3 Controllability

Consider the parallel form shown in Figure 12.6(a). To control the pole location of the closed-loop system, we are saying implicitly that the control signal, u, can control the behavior of each state variable in x. If any one of the state variables cannot be controlled by the control u, then we cannot place the poles of the system where we desire. For example, in Figure 12.6(b), if x_1 were not controllable by the control signal and if x_1 also exhibited an unstable response due to a nonzero initial condition, there would be no way to effect a state-feedback design to stabilize x_1; x_1 would perform in its own way regardless of the control signal, u. Thus, in some systems, a state-feedback design is not possible.

We now make the following definition based upon the previous discussion:

If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be *controllable;* otherwise, the system is *uncontrollable.*

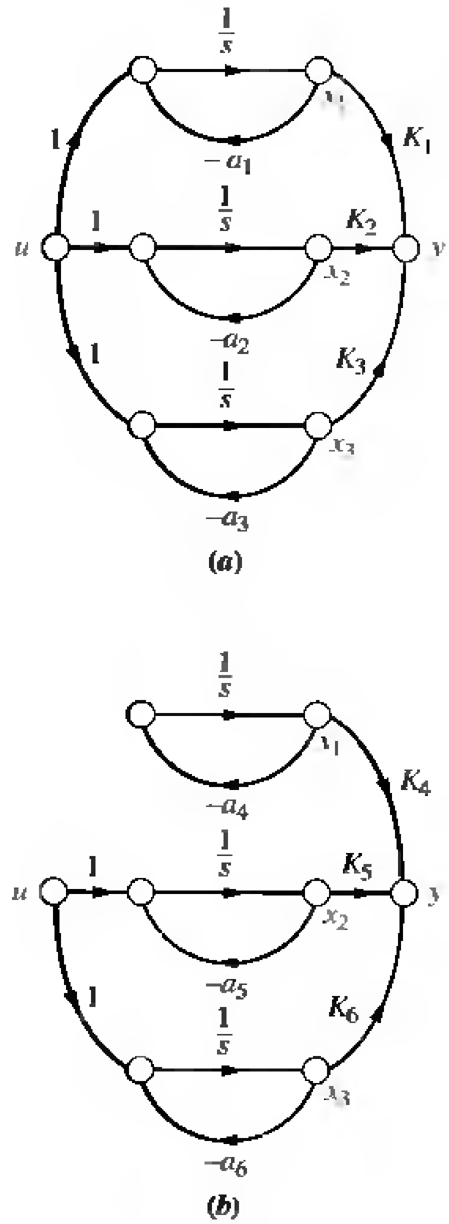


Figure 12.6 Comparison of **a.** controllable and **b.** uncontrollable systems

Pole placement is a viable design technique only for systems that are controllable. This section shows how to determine, a priori, whether pole placement is a viable design technique for a controller.

Controllability by Inspection

We can explore controllability from another viewpoint: that of the state equation itself. When the system matrix is diagonal, as it is for the parallel form, it is apparent whether or not the system is controllable. For example, the state equation for Figure 12.6(*a*) is

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u \tag{12.21}$$

or

$$\dot{x}_1 = -a_1x_1 \qquad\qquad\qquad + u \tag{12.22a}$$

$$\dot{x}_2 = \qquad\quad -a_2x_2 \qquad + u \tag{12.22b}$$

$$\dot{x}_3 = \qquad\qquad\qquad -a_3x_3 + u \tag{12.22c}$$

Since each of Eqs. (12.22) is independent and decoupled from the rest, the control u affects each of the state variables. This is controllability from another perspective.

Now let us look at the state equations for the system of Figure 12.6(*b*):

$$\dot{\mathbf{x}} = \begin{bmatrix} -a_4 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & -a_6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}u \tag{12.23}$$

or

$$\dot{x}_1 = -a_4x_1 \tag{12.24a}$$

$$\dot{x}_2 = \qquad -a_5x_2 \qquad + u \tag{12.24b}$$

$$\dot{x}_3 = \qquad\qquad\qquad -a_6x_3 + u \tag{12.24c}$$

From the state equations in (12.23) or (12.24), we see that state variable x_1 is not controlled by the control u. Thus, the system is said to be uncontrollable.

In summary, a system with distinct eigenvalues and a diagonal system matrix is controllable if the input coupling matrix **B** does not have any rows that are zero.

The Controllability Matrix

Tests for controllability that we have so far explored cannot be used for representations of the system other than the diagonal or parallel form with distinct eigenvalues. The problem of visualizing controllability gets more complicated if the system has multiple poles, even though it is represented in parallel form. Further, one cannot always determine controllability by inspection for systems that are not represented in parallel form. In other forms the existence of paths from the input to the state variables is not a criterion for controllability since the equations are not decoupled.

In order to be able to determine controllability or, alternatively, to design state feedback for a plant under any representation or choice of state variables, a matrix can be derived that must have a particular property if all state variables are to be controlled by the plant input, u. We now state the requirement for controllability, including the form, property, and name of this matrix.[2]

An nth-order plant whose state equation is

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \tag{12.25}$$

is completely controllable[3] if the matrix

$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \tag{12.26}$$

[2]See the work listed in the Bibliography by Ogata (1990: 699–702) for the derivation.

[3]*Completely controllable* means that all state variables are controllable. This textbook uses *controllable* to mean *completely controllable*.

is of rank n, where $\mathbf{C_M}$ is called the *controllability* matrix.[4] As an example, let us choose a system represented in parallel form with multiple roots.

Example 12.2

Controllability via the controllability matrix

Problem Given the system of Figure 12.7, represented by a signal-flow diagram, determine its controllability.

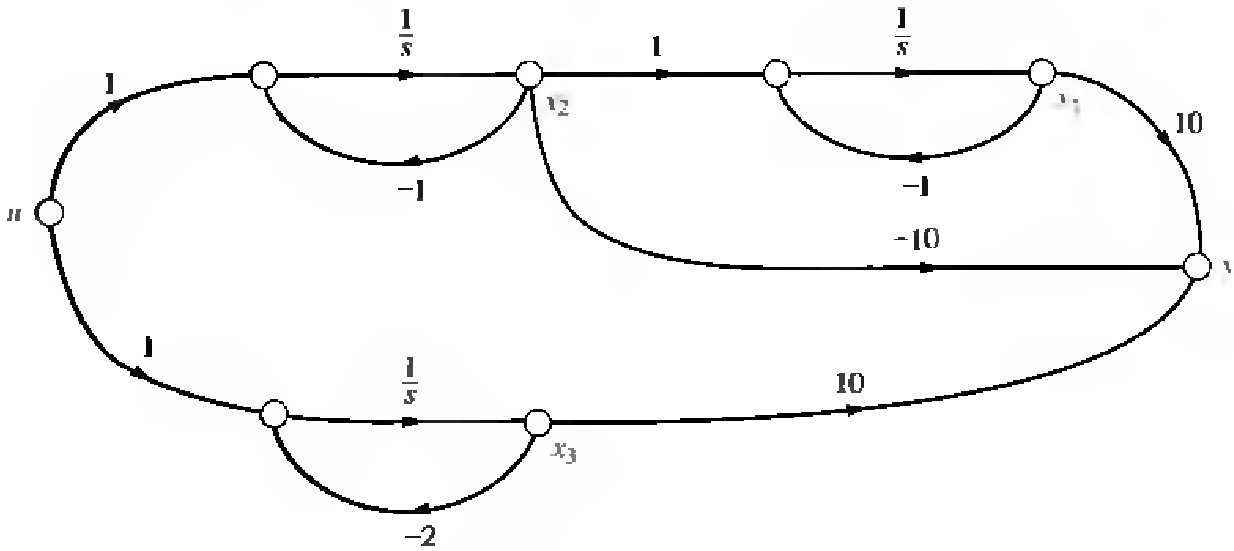


Figure 12.7 System for Example 12.2

Solution The state equation for the system written from the signal-flow diagram is

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}u \tag{12.27}$$

At first it would appear that the system is not controllable because of the zero in the **B** matrix. Remember, though, that this configuration leads to uncontrollability only if the poles are real and distinct. In this case we have multiple poles at -1.

The controllability matrix is

$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix} \tag{12.28}$$

The rank of $\mathbf{C_M}$ equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. The determinant of $\mathbf{C_M} = -1$. Since the determinant is not zero, the 3×3 matrix is nonsingular, and the rank of $\mathbf{C_M}$ is 3. We conclude that the system is controllable since the rank of $\mathbf{C_M}$ equals the system order. Thus, the poles of the system can be placed using state-variable feedback design.

[4] See Appendix F on the accompanying CD-ROM for the definition of rank. For single-input systems, instead of specifying rank n, we can say that $\mathbf{C_M}$ must be nonsingular, possess an inverse, or have linearly independent rows and columns.

MATLAB Students who are using MATLAB should now run ch12p2 in Appendix B. You will learn how to use MATLAB to test a system for controllability. This exercise solves Example 12.2 using MATLAB.

In the previous example we found that even though an element of the input coupling matrix was zero, the system was controllable. If we look at Figure 12.7, we can see why. In this figure all of the state variables are driven by the input u.

On the other hand, if we disconnect the input at either dx_1/dt, dx_2/dt, or dx_3/dt, at least one state variable would not be controllable. To see the effect, let us disconnect the input at dx_2/dt. This causes the **B** matrix to become

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{12.29}$$

We can see that the system is now uncontrollable, since x_1 and x_2 are no longer controlled by the input. This conclusion is borne out by the controllability matrix, which is now

$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & 4 \end{bmatrix} \tag{12.30}$$

Not only is the determinant of this matrix equal to zero, but so is the determinant of any 2×2 submatrix. Thus, the rank of Eq. (12.30) is 1. The system is uncontrollable because the rank of $\mathbf{C_M}$ is 1, which is less than the order, 3, of the system.

Skill-Assessment Exercise 12.2

Problem Determine whether the system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 5 \\ 0 & 3 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}u$$

is controllable.

Answer Controllable

The complete solution is on the accompanying CD-ROM.

In summary, then, pole-placement design through state-variable feedback is simplified by using the phase-variable form for the plant's state equations. However, controllability, the ability for pole-placement design to succeed, can be visualized best in the parallel form, where the system matrix is diagonal with distinct roots. In any event, the controllability matrix will always tell the designer whether the implementation is viable for state-feedback design.

The next section shows how to design state-variable feedback for systems not represented in phase-variable form. We use the controllability matrix as a tool for transforming a system to phase-variable form for the design of state-variable feedback.

12.4 Alternative Approaches to Controller Design

Section 12.2 showed how to design state-variable feedback to yield desired closed-loop poles. We demonstrated this method using systems represented in phase-variable form and saw how simple it was to calculate the feedback gains. Many times the physics of the problem requires feedback from state variables that are not phase variables. For these systems we have some choices for a design methodology.

The first method consists of matching the coefficients of det $(s\mathbf{I} - (\mathbf{A} - \mathbf{BK}))$ with the coefficients of the desired characteristic equation, which is the same method we used for systems represented in phase variables. This technique, in general, leads to difficult calculations of the feedback gains, especially for higher-order systems not represented with phase variables. Let us illustrate this technique with an example.

Example 12.3

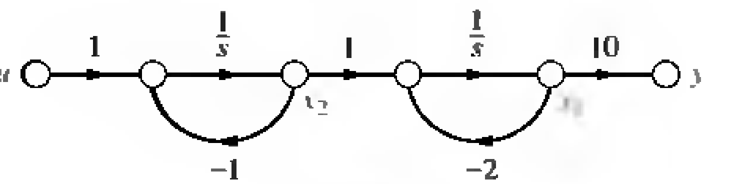
Controller design by matching coefficients

Problem Given a plant, $Y(s)/U(s) = 10/[(s+1)(s+2)]$, design state feedback for the plant represented in cascade form to yield a 15% overshoot with a settling time of 0.5 second.

Solution The signal-flow diagram for the plant in cascade form is shown in Figure 12.8(*a*). Figure 12.8(*b*) shows the system with state feedback added. Writing the state equations from Figure 12.8(*b*), we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ -k_1 & -(k_2+1) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \qquad (12.31a)$$

$$y = \begin{bmatrix} 10 & 0 \end{bmatrix}\mathbf{x} \qquad (12.31b)$$



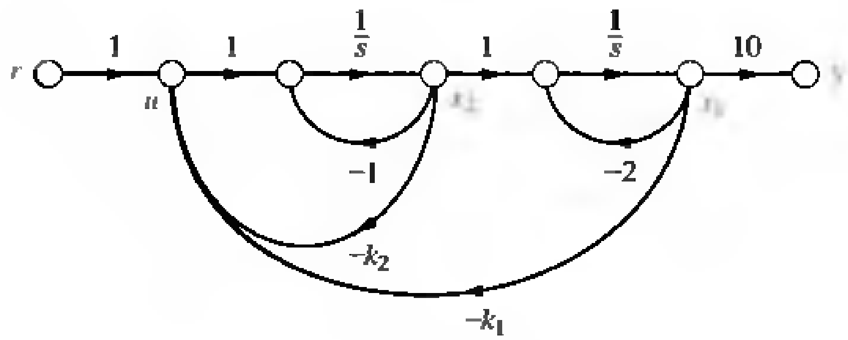


Figure 12.8 **a.** Signal-flow graph in cascade form for $G(s) = 10/[(s+1)(s+2)]$; **b.** system with state feedback added

where the characteristic equation is

$$s^2 + (k_2 + 3)s + (2k_2 + k_1 + 2) = 0 \tag{12.32}$$

Using the transient response requirements stated in the problem, we obtain the desired characteristic equation

$$s^2 + 16s + 239.5 = 0 \tag{12.33}$$

Equating the middle coefficients of Eqs. (12.32) and (12.33), we find $k_2 = 13$. Equating the last coefficients of these equations along with the result for k_2 yields $k_1 = 211.5$.

The second method consists of transforming the system to phase variables, designing the feedback gains, and transforming the designed system back to its original state-variable representation.[5] This method requires that we first develop the transformation between a system and its representation in phase-variable form.

Assume a plant not represented in phase-variable form,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{12.34a}$$

$$y = \mathbf{C}\mathbf{z} \tag{12.34b}$$

whose controllability matrix is

$$\mathbf{C}_{\mathbf{Mz}} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}] \tag{12.35}$$

Assume that the system can be transformed into the phase-variable ($\mathbf{x}$) representation with the transformation

$$\mathbf{z} = \mathbf{P}\mathbf{x} \tag{12.36}$$

Substituting this transformation into Eq. (12.34), we get

$$\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}u \tag{12.37a}$$

$$y = \mathbf{C}\mathbf{P}\mathbf{x} \tag{12.37b}$$

whose controllability matrix is

$$\begin{aligned}\mathbf{C}_{\mathbf{Mx}} &= [\mathbf{P}^{-1}\mathbf{B} \quad (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{B}) \quad (\mathbf{P}^{-1}\mathbf{AP})^2(\mathbf{P}^{-1}\mathbf{B}) \quad \cdots \quad (\mathbf{P}^{-1}\mathbf{AP})^{n-1}(\mathbf{P}^{-1}\mathbf{B})] \\ &= [\mathbf{P}^{-1}\mathbf{B} \quad (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{B}) \quad (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{B}) \quad \cdots \quad (\mathbf{P}^{-1}\mathbf{AP}) \\ &\qquad (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) \quad \cdots \quad (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{B})] \\ &= \mathbf{P}^{-1}[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}]\end{aligned} \tag{12.38}$$

Substituting Eq. (12.35) into (12.38) and solving for $\mathbf{P}$, we obtain

$$\mathbf{P} = \mathbf{C}_{\mathbf{Mz}}\mathbf{C}_{\mathbf{Mx}}^{-1} \tag{12.39}$$

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[5] See the discussions of Ackermann's formula in Franklin (1994) and Ogata (1990), listed in the Bibliography.

Thus, the transformation matrix, $\mathbf{P}$, can be found from the two controllability matrices.

After transforming the system to phase variables, we design the feedback gains as in Section 12.2. Hence, including both feedback and input, $u = -\mathbf{K_x x} + r$, Eq. (12.37) becomes

$$\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{APx} - \mathbf{P}^{-1}\mathbf{BK_x x} + \mathbf{P}^{-1}\mathbf{B}r$$

$$= (\mathbf{P}^{-1}\mathbf{AP} - \mathbf{P}^{-1}\mathbf{BK_x})\mathbf{x} + \mathbf{P}^{-1}\mathbf{B}r \tag{12.40a}$$

$$y = \mathbf{CPx} \tag{12.40b}$$

Since this equation is in phase-variable form, the zeros of this closed-loop system are determined from the polynomial formed from the elements of **CP**, as explained in Section 12.2.

Using $\mathbf{x} = \mathbf{P}^{-1}\mathbf{z}$, we transform Eq. (12.40) from phase variables back to the original representation and get

$$\dot{\mathbf{z}} = \mathbf{Az} - \mathbf{BK_x P}^{-1}\mathbf{z} + \mathbf{B}r = (\mathbf{A} - \mathbf{BK_x P}^{-1})\mathbf{z} + \mathbf{B}r \tag{12.41a}$$

$$y = \mathbf{Cz} \tag{12.41b}$$

Comparing Eq. (12.41) with (12.3), the state variable feedback gain, $\mathbf{K_z}$, for the original system is

$$\mathbf{K_z} = \mathbf{K_x P}^{-1} \tag{12.42}$$

The transfer function of this closed-loop system is the same as the transfer function for Eq. (12.40), since Eqs. (12.40) and (12.41) represent the same system. Thus, the zeros of the closed-loop transfer function are the same as the zeros of the uncompensated plant, based upon the development in Section 12.2. Let us demonstrate with a design example.

Example 12.4

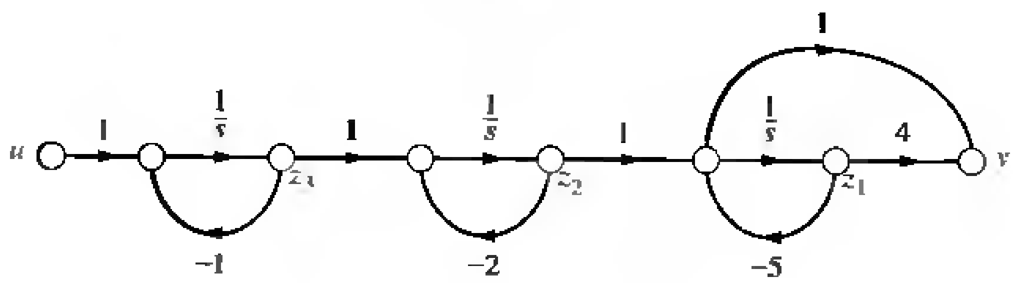
Controller design by transformation

Problem Design a state-variable feedback controller to yield a 20.8% overshoot and a settling time of 4 seconds for a plant,

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} \tag{12.43}$$

that is represented in cascade form as shown in Figure 12.9.

Figure 12.9 Signal-flow graph for plant of Example 12.4



Solution First find the state equations and the controllability matrix. The state equations written from Fignre 12.9 are

$$\dot{\mathbf{z}} = \mathbf{A_z z} + \mathbf{B_z} u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{12.44}$$

$$y = \mathbf{C_z z} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \mathbf{z}$$

from which the controllability matrix is evaluated as

$$\mathbf{C_{Mz}} = [\mathbf{B_z} \quad \mathbf{A_z B_z} \quad \mathbf{A_z^2 B_z}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -1 & 1 \end{bmatrix} \tag{12.45}$$

Since the determinant of $\mathbf{C_{Mz}}$ is -1, the system is controllable.

We now convert the system to phase variables by first finding the characteristic equation and using this equation to write the phase-variable form. The characteristic equation, $\det(s\mathbf{I} - \mathbf{A_z})$, is

$$\det(s\mathbf{I} - \mathbf{A_z}) = s^3 + 8s^2 + 17s + 10 = 0 \tag{12.46}$$

Using the coefficients of Eq. (12.46) and our knowledge of the phase-variable form, we write the phase-variable representation of the system as

$$\dot{\mathbf{x}} = \mathbf{A_x x} + \mathbf{B_z} u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{12.47a}$$

$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} \mathbf{x} \tag{12.47b}$$

The output equation was written using the coefficients of the numerator of Eq. (12.43), since the transfer function must be the same for the two representations. The controllability matrix, $\mathbf{C_{Mz}}$, for the phase-variable system is

$$\mathbf{C_{Mx}} = [\mathbf{B_x} \quad \mathbf{A_x B_z} \quad \mathbf{A_x^2 B_x}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -8 \\ 1 & -8 & 47 \end{bmatrix} \tag{12.48}$$

Using Eq. (12.39), we can now calculate the transformation matrix between the two systems as

$$\mathbf{P} = \mathbf{C_{Mz} C_{Mx}^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 10 & 7 & 1 \end{bmatrix} \tag{12.49}$$

We now design the controller using the phase-variable representation and then use Eq. (12.49) to transform the design back to the original representation. For a 20.8% overshoot and a settling time of 4 seconds, a factor of the characteristic equation of the designed closed-loop system is $s^2 + 2s + 5$. Since the closed-loop zero will be at $s = -4$, we choose the third closed-loop pole to cancel the closed-loop zero. Hence, the total characteristic equation of the desired closed-loop system is

$$D(s) = (s+4)(s^2+2s+5) = s^3+6s^2+13s+20 = 0 \tag{12.50}$$

The state equations for the phase-variable form with state-variable feedback are

$$\dot{\mathbf{x}} = (\mathbf{A_x} - \mathbf{B_x K_x})\mathbf{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(10+k_{1_x}) & -(17+k_{2_x}) & -(8+k_{3_x}) \end{bmatrix}\mathbf{x} \tag{12.51a}$$

$$y = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix}\mathbf{x} \tag{12.51b}$$

The characteristic equation for Eq. (12.51) is

$$\det(s\mathbf{I} - (\mathbf{A_x} - \mathbf{B_z K_x})) = s^3 + (8+k_{3_x})s^2 + (17+k_{2_x})s + (10+k_{1_x}) = 0 \tag{12.52}$$

Comparing Eq. (12.50) with (12.52), we see that

$$\mathbf{K_x} = \begin{bmatrix} k_{1_x} & k_{2_x} & k_{3_x} \end{bmatrix} = \begin{bmatrix} 10 & -4 & -2 \end{bmatrix} \tag{12.53}$$

Using Eqs. (12.42) and (12.49), we can transform the controller back to the original system as

$$\mathbf{K_z} = \mathbf{K_x P}^{-1} = \begin{bmatrix} -20 & 10 & -2 \end{bmatrix} \tag{12.54}$$

The final closed-loop system with state-variable feedback is shown in Figure 12.10, with the input applied as shown.

Let us now verify our design. The state equations for the designed system shown in Figure 12.10 with input r are

$$\dot{\mathbf{z}} = (\mathbf{A_z} - \mathbf{B_z K_z})\mathbf{z} + \mathbf{B_z}r = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 20 & -10 & 1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \tag{12.55a}$$

$$y = \mathbf{C_z z} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\mathbf{z} \tag{12.55b}$$

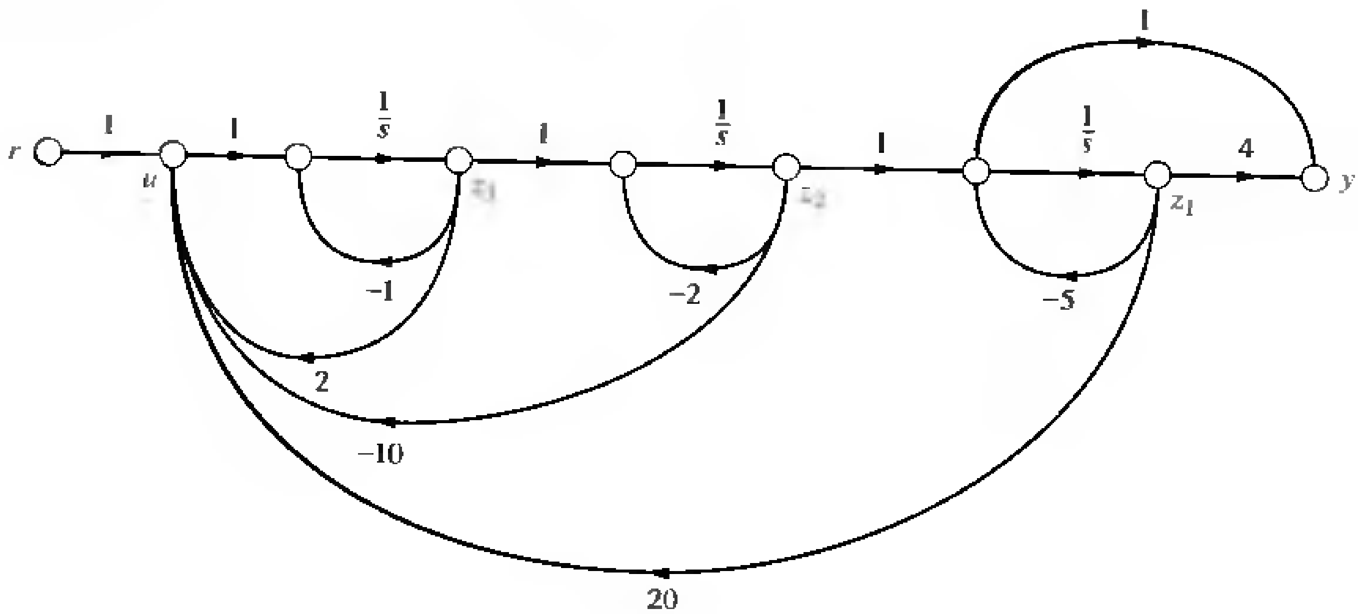


Figure 12.10 Designed system with state-variable feedback for Example 12.4

Using Eq. (3.73) to find the closed-loop transfer function, we obtain

$$T(s) = \frac{(s+4)}{s^3+6s^2+13s+20} = \frac{1}{s^2+2s+5} \tag{12.56}$$

The requirements for our design have been met.

MATLAB

Students who are using MATLAB should now run ch12p3 in Appendix B. You will learn how to use MATLAB to design a controller for a plant not represented in phase-variable form. You will see that MATLAB does not require transformation to phase-variable form. This exercise solves Example 12.4 using MATLAB.

Skill-Assessment Exercise 12.3

Control Solution

Problem Design a linear state-feedback controller to yield 20% overshoot and a settling time of 2 seconds for a plant,

$$G(s) = \frac{(s+6)}{(s+9)(s+8)(s+7)}$$

that is represented in state space in cascade form by

$$\dot{\mathbf{z}} = \mathbf{Az} + \mathbf{B}u = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & -9 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{Cz} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\mathbf{z}$$

Answer $\mathbf{K_z} = [-40.23 \quad 62.24 \quad -14]$

The complete solution is on the accompanying CD-ROM.

In this section we saw how to design state-variable feedback for plants not represented in phase-variable form. Using controllability matrices, we were able to transform a plant to phase-variable form, design the controller, and finally transform the controller design back to the plant's original representation. The design of the controller relies on the availability of the states for feedback. In the next section we discuss the design of state-variable feedback when some or all of the states are not available.

12.5 Observer Design

Controller design relies upon access to the state variables for feedback through adjustable gains. This access can be provided by hardware. For example, gyros can measure position and velocity on a space vehicle. Sometimes it is impractical to use this hardware for reasons of cost, accuracy, or availability. For example, in powered flight of space vehicles, inertial measuring units can be used to

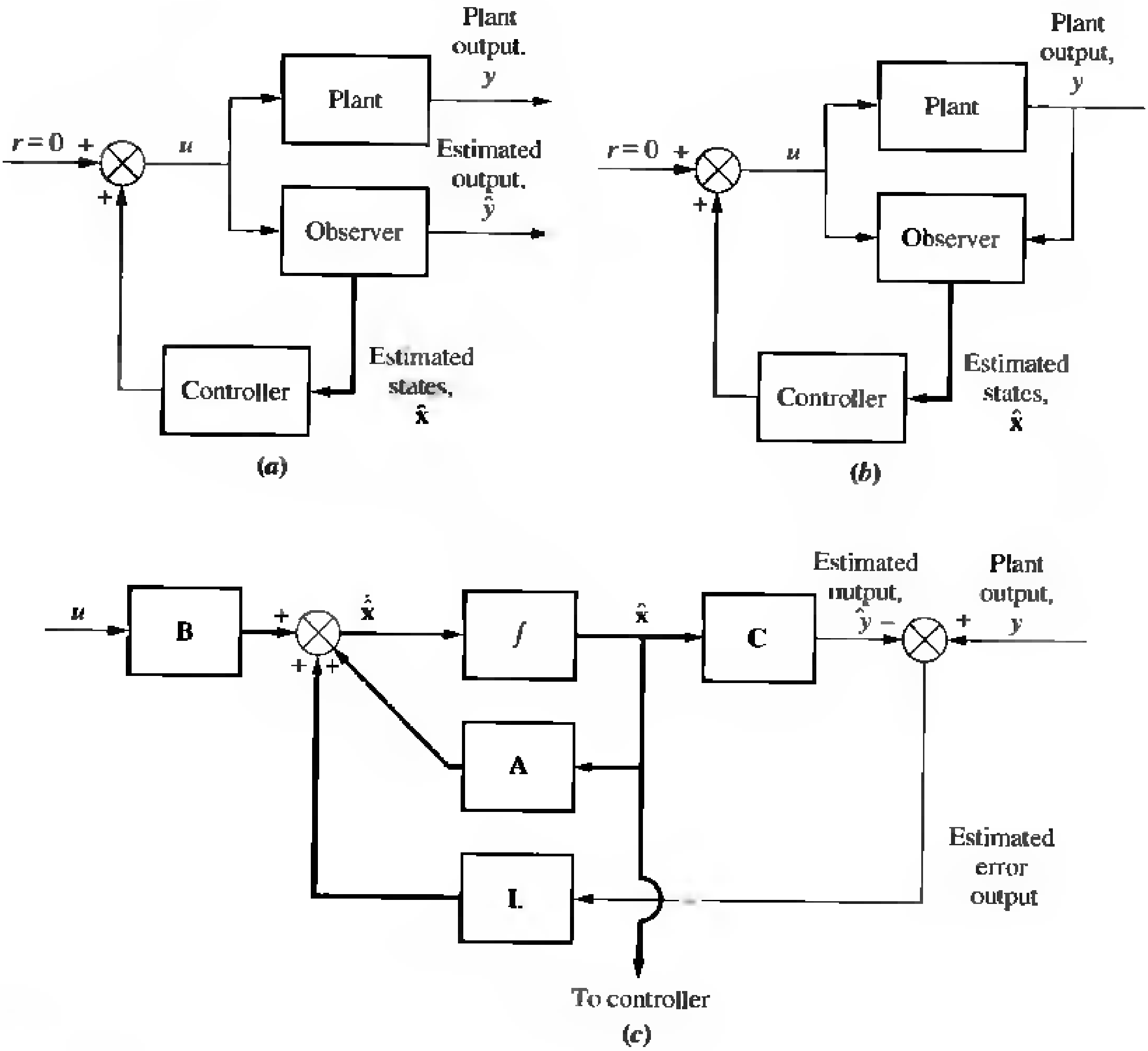


Figure 12.11 State-feedback design using an observer to estimate unavailable state variables: **a.** open-loop observer; **b.** closed-loop observer; **c.** exploded view of a closed-loop observer, showing feedback arrangement to reduce state-variable estimation error

calculate the acceleration. However, their alignment deteriorates with time; thus, other means of measuring acceleration may be desirable (Rockwell International, 1984). In other applications, some of the state variables may not be available at all, or it is too costly to measure them or send them to the controller. If the state variables are not available because of system configuration or cost, it is possible to estimate the states. Estimated states, rather than actual states, are then fed to the controller. One scheme is shown in Figure 12.11(*a*). An *observer,* sometimes called an *estimator,* is used to calculate state variables that are not accessible from the plant. Here the observer is a model of the plant.

Let us look at the disadvantages of such a configuration. Assume a plant,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{12.57a}$$

$$y = \mathbf{C}\mathbf{x} \tag{12.57b}$$

and an observer.

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u \tag{12.58a}$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}} \tag{12.58b}$$

Subtracting Eqs. (12.58) from (12.57), we obtain

$$\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) \tag{12.59a}$$

$$y - \hat{y} = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) \tag{12.59b}$$

Thus, the dynamics of the difference between the actual and estimated states is unforced, and if the plant is stable, this difference, due to differences in initial state vectors, approaches zero. However, the speed of convergence between the actual state and the estimated state is the same as the transient response of the plant since the characteristic equation for (12.59*a*) is the same as for (12.57*a*). Since the convergence is too slow, we seek a way to speed up the observer and make its response time much faster than that of the controlled closed-loop system, so that, effectively, the controller will receive the estimated states instantaneously.

To increase the speed of convergence between the actual and estimated states, we use feedback, shown conceptually in Figure 12.11(*b*) and in more detail in Figure 12.11(*c*). The error between the outputs of the plant and the observer is fed back to the derivatives of the observer's states. The system corrects to drive this error to zero. With feedback we can design a desired transient response into the observer that is much quicker than that of the plant or controlled closed-loop system.

When we implemented the controller, we found that the phase-variable or controller canonical form yielded an easy solution for the controller gains. In designing an observer, it is the observer canonical form that yields the easy solution for the observer gains. Figure 12.12(*a*) shows an example of a third-order plant represented in observer canonical form. In Figure 12.12(*b*) the plant is configured as an observer with the addition of feedback, as previously described.

The design of the observer is separate from the design of the controller. Similar to the design of the controller vector, **K**, the design of the observer consists of evaluating the constant vector, **L**, so that the transient response of the observer is faster than the response of the controlled loop in order to yield a rapidly updated estimate of the state vector. We now derive the design methodology.

We will first find the state equations for the error between the actual state vector and the estimated state vector, $(\mathbf{x} - \hat{\mathbf{x}})$. Then we will find the characteristic equation for the error system and evaluate the required **L** to meet a rapid transient response for the observer.

Writing the state equations of the observer from Figure 12.11(*c*), we have

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \hat{y}) \tag{12.60a}$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}} \tag{12.60b}$$

But the state equations for the plant are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{12.61a}$$

$$y = \mathbf{C}\mathbf{x} \tag{12.61b}$$

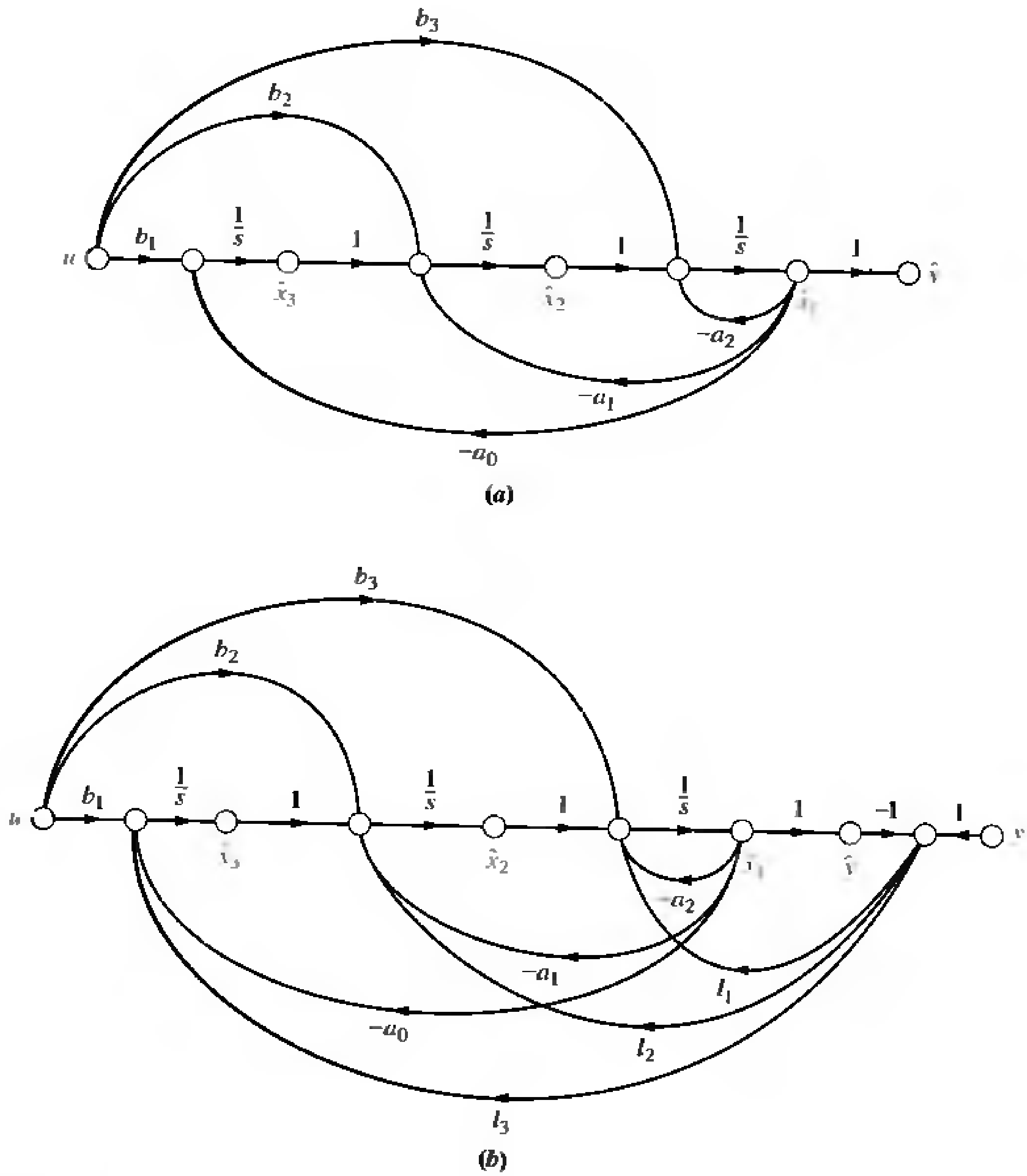


Figure 12.12
Third-order observer in observer canonical form.
a. before the addition of feedback;
b. after the addition of feedback

Subtracting Eqs. (12.60) from (12.61), we obtain

$$(\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) - \mathbf{L}(y - \hat{y}) \tag{12.62a}$$

$$(y - \hat{y}) = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) \tag{12.62b}$$

where $\mathbf{x} - \hat{\mathbf{x}}$ is the error between the actual state vector and the estimated state vector, and $y - \hat{y}$ is the error between the actual output and the estimated output.

Substituting the output equation into the state equation, we obtain the state equation for the error between the estimated state vector and the actual state vector:

$$(\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}}) = (\mathbf{A} - \mathbf{LC})(\mathbf{x} - \hat{\mathbf{x}}) \tag{12.63a}$$

$$(y - \hat{y}) = \mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) \tag{12.63b}$$

Letting $\mathbf{e_x} = (\mathbf{x} - \hat{\mathbf{x}})$, we have

$$\dot{\mathbf{e}}_\mathbf{x} = (\mathbf{A} - \mathbf{LC})\mathbf{e_x} \tag{12.64a}$$

$$y - \hat{y} = \mathbf{Ce_x} \tag{12.64b}$$

Equation (12.64*a*) is unforced. If the eigenvalues are all negative, the estimated state vector error, $\mathbf{e_x}$, will decay to zero. The design then consists of solving for the values of $\mathbf{L}$ to yield a desired characteristic equation or response for Eqs. (12.64). The characteristic equation is found from Eqs. (12.64) to be

$$\det[\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})] = 0 \tag{12.65}$$

Now we select the eigenvalues of the observer to yield stability and a desired transient response that is faster than the controlled closed-loop response. These eigenvalues determine a characteristic equation that we set equal to Eq. (12.65) to solve for $\mathbf{L}$.

Let us demonstrate the procedure for an nth-order plant represented in observer canonical form. We first evaluate $\mathbf{A} - \mathbf{LC}$. The form of $\mathbf{A}$, $\mathbf{L}$, and $\mathbf{C}$ can be derived by extrapolating the form of these matrices from a third-order plant, which you can derive from Figure 12.12. Thus,

$$\mathbf{A} - \mathbf{LC} = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{bmatrix} [1 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0]$$

$$= \begin{bmatrix} -(a_{n-1} + l_1) & 1 & 0 & 0 & \cdots & 0 \\ -(a_{n-2} + l_2) & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_1 + l_{n-1}) & 0 & 0 & 0 & \cdots & 1 \\ -(a_0 + l_n) & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{12.66}$$

The characteristic equation for $\mathbf{A} - \mathbf{LC}$ is

$$s^n + (a_{n-1} + l_1)s^{n-1} + (a_{n-2} + l_2)s^{n-2} + \cdots + (a_1 + l_{n-1})s + (a_0 + l_n) = 0 \tag{12.67}$$

Notice the relationship between Eq. (12.67) and the characteristic equation, $\det(s\mathbf{I} - \mathbf{A}) = 0$, for the plant, which is

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0 = 0 \tag{12.68}$$

Thus, if desired, Eq. (12.67) can be written by inspection if the plant is represented in observer canonical form. We now equate Eq. (12.67) with the desired closed-loop observer characteristic equation, which is chosen on the basis of a desired transient response. Assume the desired characteristic equation is

$$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1 s + d_0 = 0 \tag{12.69}$$

We can now solve for the l_i's by equating the coefficients of Eqs. (12.67) and (12.69):

$$l_i = d_{n-i} - a_{n-i} \qquad i = 1, 2, \ldots, n \tag{12.70}$$

Let us demonstrate the design of an observer using the observer canonical form. In subsequent sections we will show how to design the observer for other than observer canonical form.

Example 12.5

Observer design for observer canonical form

Problem Design an observer for the plant

$$G(s) = \frac{(s+4)}{(s+1)(s+2)(s+5)} = \frac{s+4}{s^3 + 8s^2 + 17s + 10} \tag{12.71}$$

which is represented in observer canonical form. The observer will respond 10 times faster than the controlled loop designed in Example 12.4.

Solution

1. First represent the estimated plant in observer canonical form. The result is shown in Figure 12.13(*a*).
2. Now form the difference between the plant's actual output, y, and the observer's estimated output, $\hat{y}$, and add the feedback paths from this difference to the derivative of each state variable. The result is shown in Figure 12.13(*b*).
3. Next find the characteristic polynomial. The state equations for the estimated plant shown in Figure 12.13(*a*) are

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}u \tag{12.72a}$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\hat{\mathbf{x}} \tag{12.72b}$$

From Eqs. (12.64) and (12.66), the observer error is

$$\dot{\mathbf{e}}_x = (\mathbf{A} - \mathbf{LC})\mathbf{e}_x = \begin{bmatrix} -(8+l_1) & 1 & 0 \\ -(17+l_2) & 0 & 1 \\ -(10+l_3) & 0 & 0 \end{bmatrix}\mathbf{e}_x \tag{12.73}$$

Using Eq. (12.65), we obtain the characteristic polynomial

$$s^3 + (8+l_1)s^2 + (17+l_2)s + (10+l_3) \tag{12.74}$$

4. Now evaluate the desired polynomial, set the coefficients equal to those of Eq. (12.74), and solve for the gains, l_i. From Eq. (12.50), the closed-loop

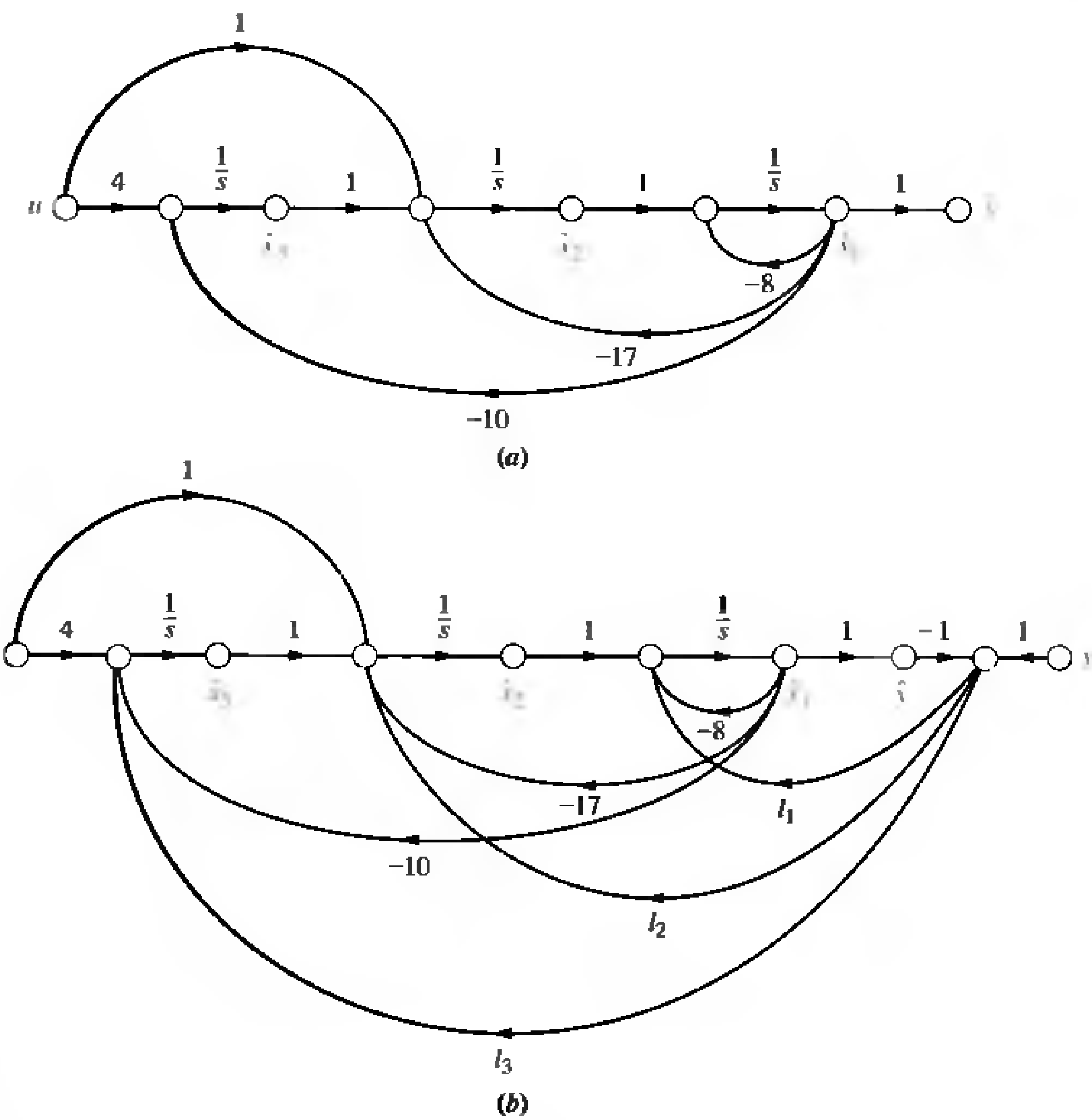


Figure 12.13
a. Signal-flow graph of a system using observer canonical form variables;
b. additional feedback to create observer

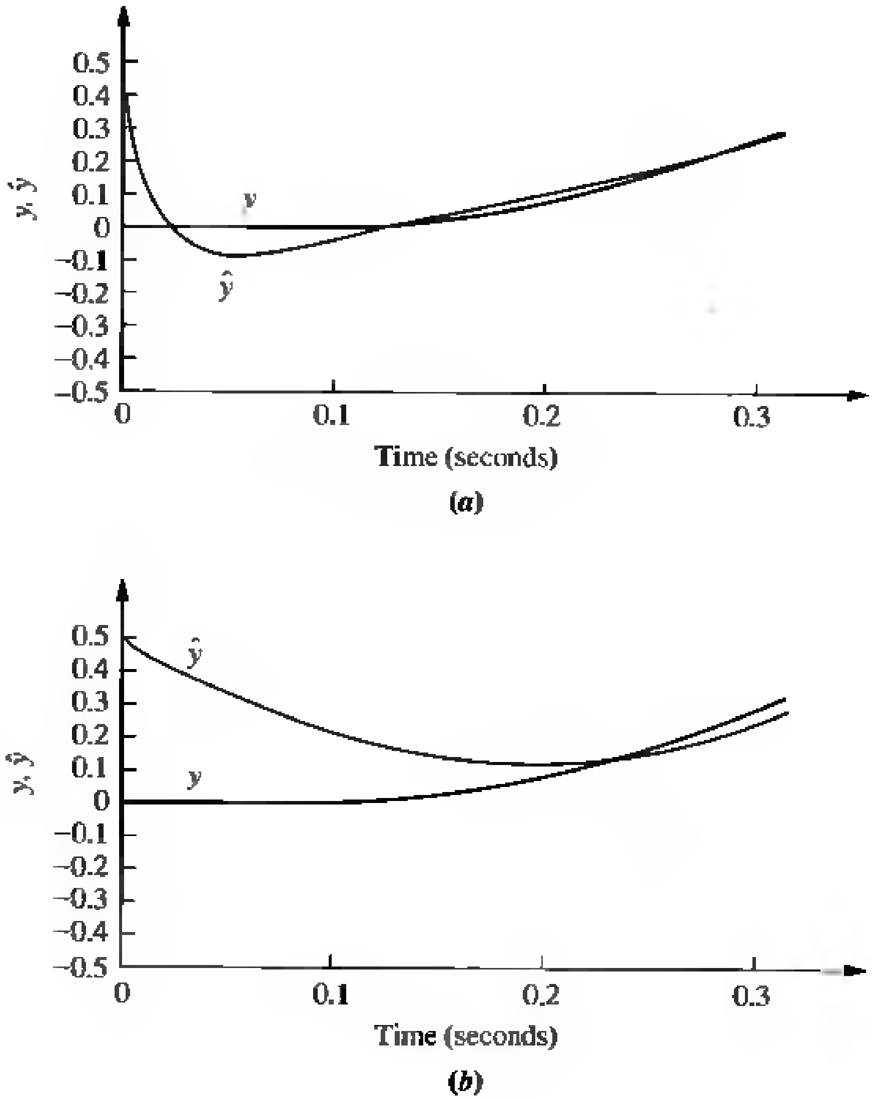
controlled system has dominant second-order poles at $-1 \pm j2$. To make our observer 10 times faster, we design the observer poles to be at $-10 \pm j20$. We select the third pole to be 10 times the real part of the dominant second-order poles, or -100. Hence, the desired characteristic polynomial is

$$(s + 100)(s^2 + 20s + 500) = s^3 + 120s^2 + 2500s + 50{,}000 \quad (12.75)$$

Equating Eqs. (12.74) and (12.75), we find $l_1 = 112$, $l_2 = 2483$, and $l_3 = 49{,}990$.

A simulation of the observer with an input of $r(t) = 100t$ is shown in Figure 12.14. The initial conditions of the plant were all zero, and the initial condition of $\hat{x}_1$ was 0.5. Since the dominant pole of the observer is $-10 \pm j20$, the expected settling time should be about 0.4 second. It is interesting to note the slower response in Figure 12.14(*b*), where the observer gains are disconnected, and the observer is simply a copy of the plant with a different initial condition.

Figure 12.14 Simulation showing response of observer: **a.** closed-loop; **b.** open-loop with observer gains disconnected



MATLAB Students who are using MATLAB should now run ch12p4 in Appendix B. You will learn how to use MATLAB to design an observer using pole placement. This exercise solves Example 12.5 using MATLAB.

Skill-Assessment Exercise 12.4

Control Solution

Problem Design an observer for the plant

$$G(s) = \frac{(s+6)}{(s+7)(s+8)(s+9)}$$

whose estimated plant is represented in state space in observer canonical form as

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u = \begin{bmatrix} -24 & 1 & 0 \\ -191 & 0 & 1 \\ -504 & 0 & 0 \end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}u$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\hat{\mathbf{x}}$$

The observer will respond 10 times faster than the controlled loop designed in in Skill-Assessment Exercise 12.3.

Answer $\mathbf{L} = [216 \quad 9730 \quad 383{,}696]^T$, where T signifies vector transpose.

The complete solution is on the accompanying CD-ROM.

In this section we designed an observer in observer canonical form that uses the output of a system to estimate the state variables. In the next section we examine the conditions under which an observer cannot be designed.

12.6 Observability

Recall that the ability to control all of the state variables is a requirement for the design of a controller. State-variable feedback gains cannot be designed if any state variable is uncontrollable. Uncontrollability can be viewed best with diagonalized systems. The signal-flow graph showed clearly that the uncontrollable state variable was not connected to the control signal of the system.

A similar concept governs our ability to create a design for an observer. Specifically, we are using the output of a system to deduce the state variables. If any state variable has no effect upon the output, then we cannot evaluate this state variable by observing the output.

The ability to observe a state variable from the output is best seen from the diagonalized system. Figure 12.15(*a*) shows a system where each state variable can be observed at the output since each is connected to the output. Figure 12.15(*b*) is an example of a system where all state variables cannot be observed at the output. Here x_1 is not connected to the output and could not be estimated from a measurement of the output.

We now make the following definition based upon the previous discussion:

> If the initial-state vector, $\mathbf{x}(t_0)$, can be found from $u(t)$ and $y(t)$ measured over a finite interval of time from t_0, the system is said to be *observable*; otherwise the system is said to be *unobservable*.

Simply stated, observability is the ability to deduce the state variables from a knowledge of the input, $u(t)$, and the output, $y(t)$. Pole placement for an observer is a viable design technique only for systems that are observable. This section shows how to determine, a priori, whether or not pole placement is a viable design technique for an observer.

Observability by Inspection

We can also explore observability from the output equation of a diagonalized system. The output equation for the diagonalized system of Figure 12.15(*a*) is

$$y = \mathbf{C}\mathbf{x} = [1 \quad 1 \quad 1]\mathbf{x} \tag{12.76}$$

On the other hand, the output equation for the unobservable system of Figure 12.15(*b*) is

$$y = \mathbf{C}\mathbf{x} = [0 \quad 1 \quad 1]\mathbf{x} \tag{12.77}$$

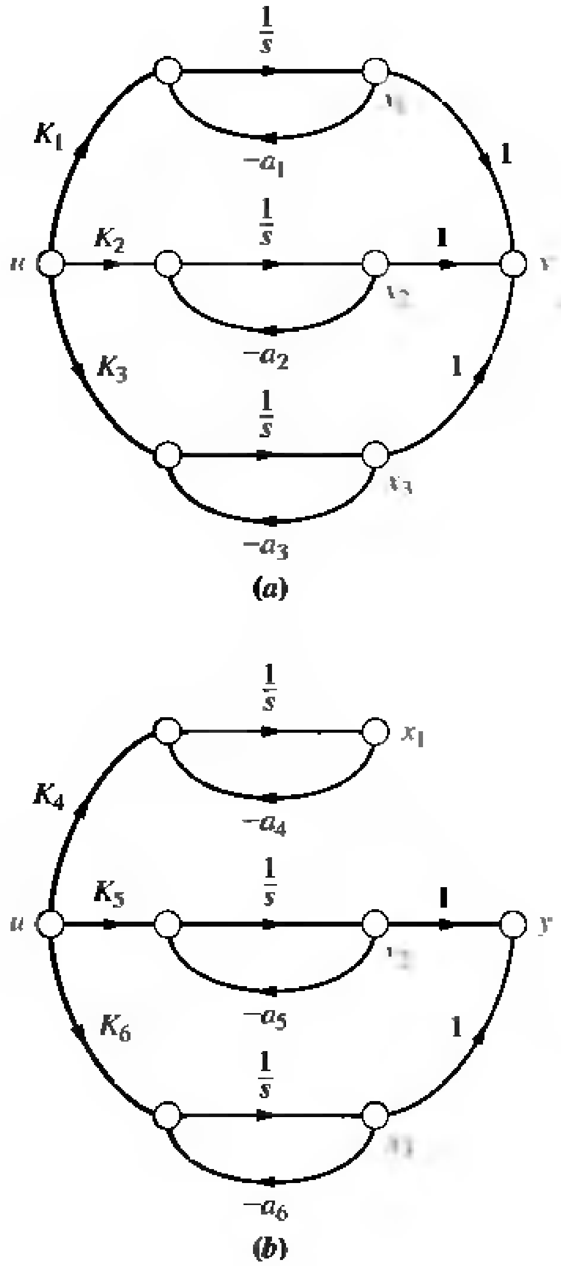


Figure 12.15 Comparison of **a.** observable and **b.** unobservable systems

Notice that the first column of Eq. (12.77) is zero. For systems represented in parallel form with distinct eigenvalues, if any column of the output coupling matrix is zero, the diagonal system is not observable.

The Observability Matrix

Again, as for controllability, systems represented in other than diagonalized form cannot be reliably evaluated for observability by inspection. In order to determine observability for systems under any representation or choice of state variables, a matrix can be derived that must have a particular property if all state variables are to be observed at the output. We now state the requirements for observability, including the form, property, and name of this matrix.

An nth-order plant whose state and output equations are, respectively,

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{12.78a}$$

$$\mathbf{y} = \mathbf{Cx} \tag{12.78b}$$

is completely observable[6] if the matrix

$$\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{12.79}$$

is of rank n, where $\mathbf{O}_M$ is called the *observability matrix*.[7]

The following two examples illustrate the use of the observability matrix.

Example 12.6

Observability via the observability matrix

Problem Determine if the system of Figure 12.16 is observable.

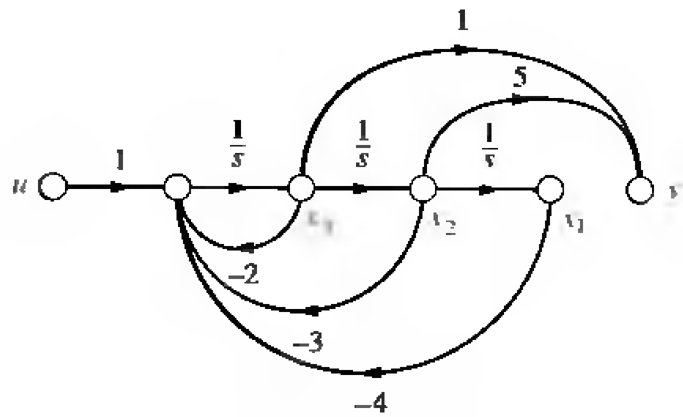


Figure 12.16 System of Example 12.6

Solution The state and output equations for the system are

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \tag{12.80a}$$

$$y = \mathbf{Cx} = [0 \quad 5 \quad 1]\mathbf{x} \tag{12.80b}$$

[6]*Completely observable* means that all state variables are observable. This textbook uses *observable* to mean *completely observable*.

[7]See Ogata (1990: 706–708) for a derivation.

Thus, the observability matrix, $\mathbf{O_M}$, is

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ -4 & -3 & 3 \\ -12 & -13 & -9 \end{bmatrix} \tag{12.81}$$

Since the determinant of $\mathbf{O_M}$ equals -344, $\mathbf{O_M}$ is of full rank equal to 3. The system is thus observable.

You might have been misled and concluded by inspection that the system is unobservable because the state variable x_1 is not fed *directly* to the output. Remember that conclusions about observability by inspection are valid only for diagonalized systems that have distinct eigenvalues.

MATLAB

Students who are using MATLAB should now run ch12p5 in Appendix B. You will learn how to use MATLAB to test a system for observability. This exercise solves Example 12.6 using MATLAB.

Example 12.7

Unobservability via the observability matrix

Problem Determine whether the system of Figure 12.17 is observable.

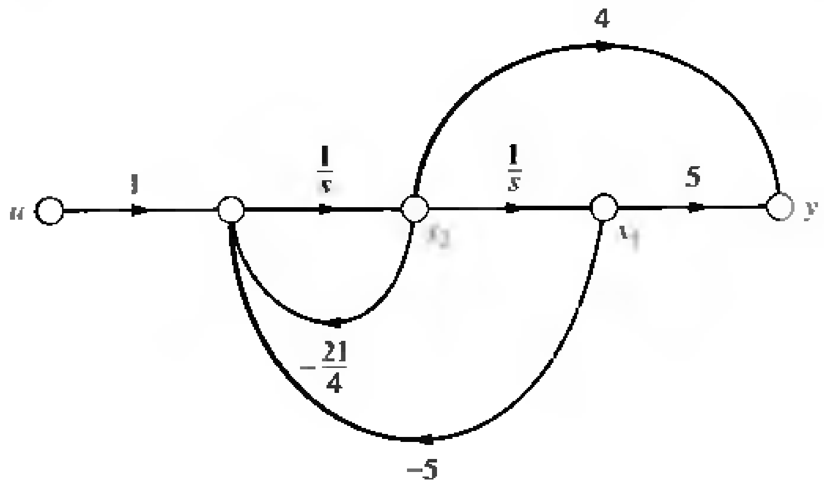


Figure 12.17 System of Example 12.7

Solution The state and output equations for the system are

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u = \begin{bmatrix} 0 & 1 \\ -5 & -21/4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \tag{12.82a}$$

$$y = \mathbf{Cx} = [5 \quad 4]\mathbf{x} \tag{12.82b}$$

The observability matrix, $\mathbf{O_M}$, for this system is

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -20 & -16 \end{bmatrix} \tag{12.83}$$

The determinant for this observability matrix equals 0. Thus, the observability matrix does not have full rank, and the system is not observable.

Again, you might conclude by inspection that the system is observable because all states feed the output. Remember that observability by inspection is valid only for a diagonalized representation of a system with distinct eigenvalues.

Skill-Assessment Exercise 12.5

Problem Determine whether the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -2 & -1 & -3 \\ 0 & -2 & 1 \\ -7 & -8 & -9 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}u$$

$$y = \mathbf{C}\mathbf{x} = [4 \quad 6 \quad 8]\mathbf{x}$$

is observable.

Answer Observable.

The complete solution is on the accompanying CD-ROM.

Now that we have discussed observability and the observability matrix, we are ready to talk about the design of an observer for a plant not represented in observer canonical form.

12.7 Alternative Approaches to Observer Design

Earlier in the chapter we discussed how to design controllers for systems not represented in phase-variable form. One method is to match the coefficients of $\det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})]$ with the coefficients of the desired characteristic polynomial. This method can yield difficult calculations for higher-order systems. Another method is to transform the plant to phase-variable form, design the controller, and transfer the design back to its original representation. The transformations were derived from the controllability matrix.

In this section we use a similar idea for the design of observers not represented in observer canonical form. One method is to match the coefficients of $\det[s\mathbf{I} - (\mathbf{A} - \mathbf{LC})]$ with the coefficients of the desired characteristic polynomial. Again, this method can yield difficult calculations for higher-order systems. Another method is first to transform the plant to observer canonical form so that the design equations are simple, then perform the design in observer canonical form, and finally transform the design back to the original representation.

Let us pursue this second method. First we will derive the transformation between a system representation and its representation in observer canonical form. Assume a plant not represented in observer canonical form,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u \qquad (12.84a)$$

$$y = \mathbf{C}\mathbf{z} \qquad (12.84b)$$

whose observability matrix is

$$\mathbf{O_{Mz}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-2} \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{12.85}$$

Now assume that the system can be transformed to the observer canonical form, $\mathbf{x}$, with the transformation

$$\mathbf{z} = \mathbf{Px} \tag{12.86}$$

Substituting Eq. (12.86) into Eq. (12.84) and premultiplying the state equation by $\mathbf{P}^{-1}$, we find that the state equations in observer canonical form are

$$\dot{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{APx} + \mathbf{P}^{-1}\mathbf{B}u \tag{12.87a}$$

$$y = \mathbf{CPx} \tag{12.87b}$$

whose observability matrix, $\mathbf{O_{Mx}}$, is

$$\mathbf{O_{Mx}} = \begin{bmatrix} \mathbf{CP} \\ \mathbf{CP}(\mathbf{P}^{-1}\mathbf{AP}) \\ \mathbf{CP}(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) \\ \vdots \\ \mathbf{CP}(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})\cdots(\mathbf{P}^{-1}\mathbf{AP}) \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}\mathbf{P} \tag{12.88}$$

Substituting Eq. (12.85) into (12.88) and solving for $\mathbf{P}$, we obtain

$$\mathbf{P} = \mathbf{O_{Mz}}^{-1}\mathbf{O_{Mx}} \tag{12.89}$$

Thus, the transformation, $\mathbf{P}$, can be found from the two observability matrices.

After transforming the plant to observer canonical form, we design the feedback gains, $\mathbf{L_x}$, as in Section 12.5. Using the matrices from Eq. (12.87) and the form suggested by Eq. (12.64), we have

$$\dot{\mathbf{e}}_\mathbf{x} = (\mathbf{P}^{-1}\mathbf{AP} - \mathbf{L_x CP})\mathbf{e_x} \tag{12.90a}$$

$$y - \hat{y} = \mathbf{CPe_x} \tag{12.90b}$$

Since $\mathbf{x} = \mathbf{P}^{-1}\mathbf{z}$, and $\hat{\mathbf{x}} = \mathbf{P}^{-1}\hat{\mathbf{z}}$, then $\mathbf{e_x} = \mathbf{x} - \hat{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{e_z}$. Substituting $\mathbf{e_x} = \mathbf{P}^{-1}\mathbf{e_x}$ into Eqs. (12.90) transforms Eqs. (12.90) back to the original representation. The result is

$$\dot{\mathbf{e}}_\mathbf{x} = (\mathbf{A} - \mathbf{PL_x C})\mathbf{e_z} \tag{12.91a}$$

$$y - \hat{y} = \mathbf{Ce_z} \tag{12.91b}$$

Comparing Eq. (12.91*a*) to (12.64*a*), we see that the observer gain vector is

$$\mathbf{L_x} = \mathbf{PL_x} \tag{12.92}$$

We now demonstrate the design of an observer for a plant not represented in observer canonical form. The first example uses transformations to and from observer canonical form. The second example matches coefficients without the transformation. This method, however, can become difficult if the system order is high.

Example 12.8

Observer design by transformation

Problem Design an observer for the plant

$$G(s) = \frac{1}{(s+1)(s+2)(s+5)} \tag{12.93}$$

represented in cascade form. The closed-loop performance of the observer is governed by the characteristic polynomial used in Example 12.5: $s^3 + 120s^2 + 2500s + 50{,}000$.

Solution First represent the plant in its original cascade form.

$$\dot{\mathbf{z}} = \mathbf{Az} + \mathbf{B}u = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \tag{12.94a}$$

$$y = \mathbf{Cz} = [1 \quad 0 \quad 0]\mathbf{z} \tag{12.94b}$$

The observability matrix, $\mathbf{O_{Mz}}$, is

$$\mathbf{O_{Mx}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 25 & -7 & 1 \end{bmatrix} \tag{12.95}$$

whose determinant equals 1. Hence, the plant is observable.

The characteristic equation for the plant is

$$\det(s\mathbf{I} - \mathbf{A}) = s^3 + 8s^2 + 17s + 10 = 0 \tag{12.96}$$

We can use the coefficients of this characteristic polynomial to form the observer canonical form:

$$\dot{\mathbf{x}} = \mathbf{A_x x} + \mathbf{B_x}u \tag{12.97a}$$

$$y = \mathbf{C_x x} \tag{12.97b}$$

where

$$\mathbf{A_x} = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix}; \qquad \mathbf{C_x} = [1 \quad 0 \quad 0] \tag{12.98}$$

The observability matrix for the observer canonical form is

$$\mathbf{O_{Mx}} = \begin{bmatrix} \mathbf{C_x} \\ \mathbf{C_xA_x} \\ \mathbf{C_xA_x}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 1 & 0 \\ 47 & -8 & 1 \end{bmatrix} \tag{12.99}$$

We now design the observer for the observer canonical form. First form $(\mathbf{A_x} - \mathbf{L_xC_x})$,

$$\mathbf{A_x} - \mathbf{L_xC_x} = \begin{bmatrix} -8 & 1 & 0 \\ -17 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} [1 \quad 0 \quad 0]$$

$$= \begin{bmatrix} -(8+l_1) & 1 & 0 \\ -(17+l_2) & 0 & 1 \\ -(10+l_3) & 0 & 0 \end{bmatrix} \tag{12.100}$$

whose characteristic polynomial is

$$\det[s\mathbf{I} - (\mathbf{A_x} - \mathbf{L_xC_x})] = s^3 + (8+l_1)s^2 + (17+l_2)s + (10+l_3) \tag{12.101}$$

Equating this polynomial to the desired closed-loop observer characteristic equation, $s^3 + 120s^2 + 2500s + 50{,}000$, we find

$$\mathbf{L_x} = \begin{bmatrix} 112 \\ 2483 \\ 49{,}990 \end{bmatrix} \tag{12.102}$$

Now transform the design back to the original representation. Using Eq. (12.89), the transformation matrix is

$$\mathbf{P} = \mathbf{O_{Mz}}^{-1}\mathbf{O_{Mx}} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \tag{12.103}$$

Transforming $\mathbf{L_x}$ to the original representation, we obtain

$$\mathbf{L_x} = \mathbf{PL_x} = \begin{bmatrix} 112 \\ 2147 \\ 47{,}619 \end{bmatrix} \tag{12.104}$$

The final configuration is shown in Figure 12.18.

A simulation of the observer is shown in Figure 12.19(a). To demonstrate the effect of the observer design, Figure 12.19(b) shows the reduced speed if the observer is simply a copy of the plant and all observer feedback paths are disconnected.

MATLAB

Students who are using MATLAB should now run ch12p6 in Appendix B. You will learn how to use MATLAB to design an observer for a plant not represented in observer canonical form. You will see that MATLAB does not require transformation to observer canonical form. This exercise solves Example 12.8 using MATLAB.

Figure 12.18
Observer design

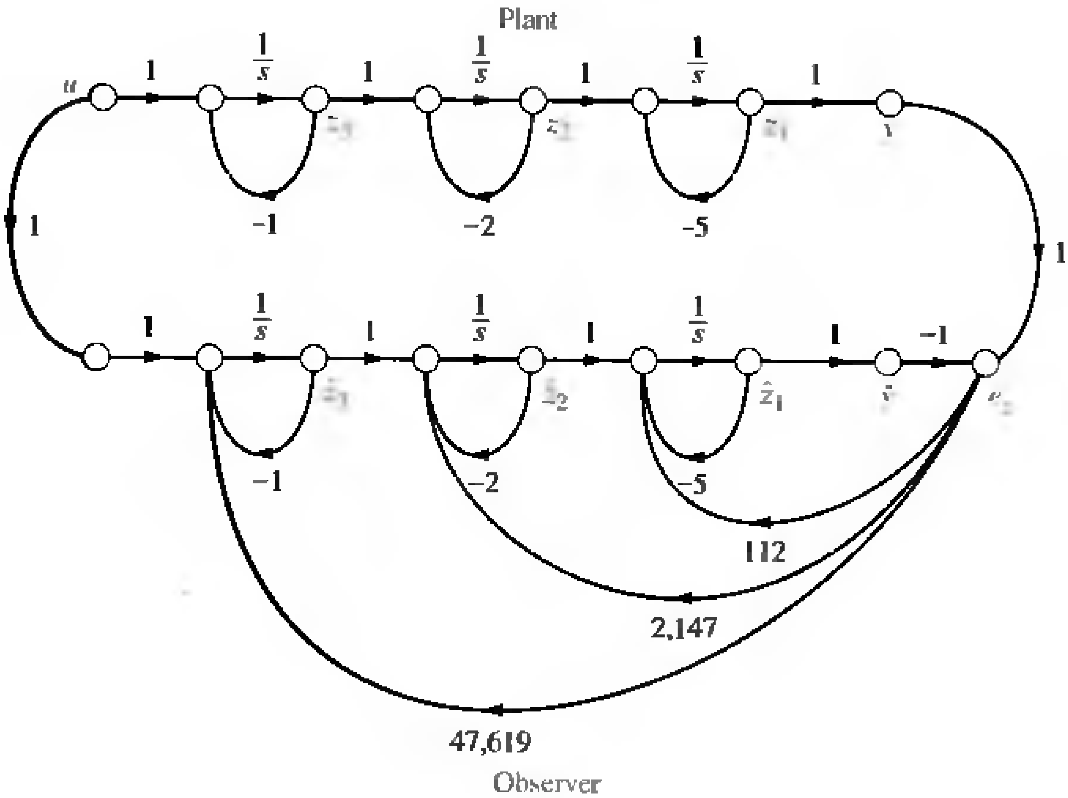
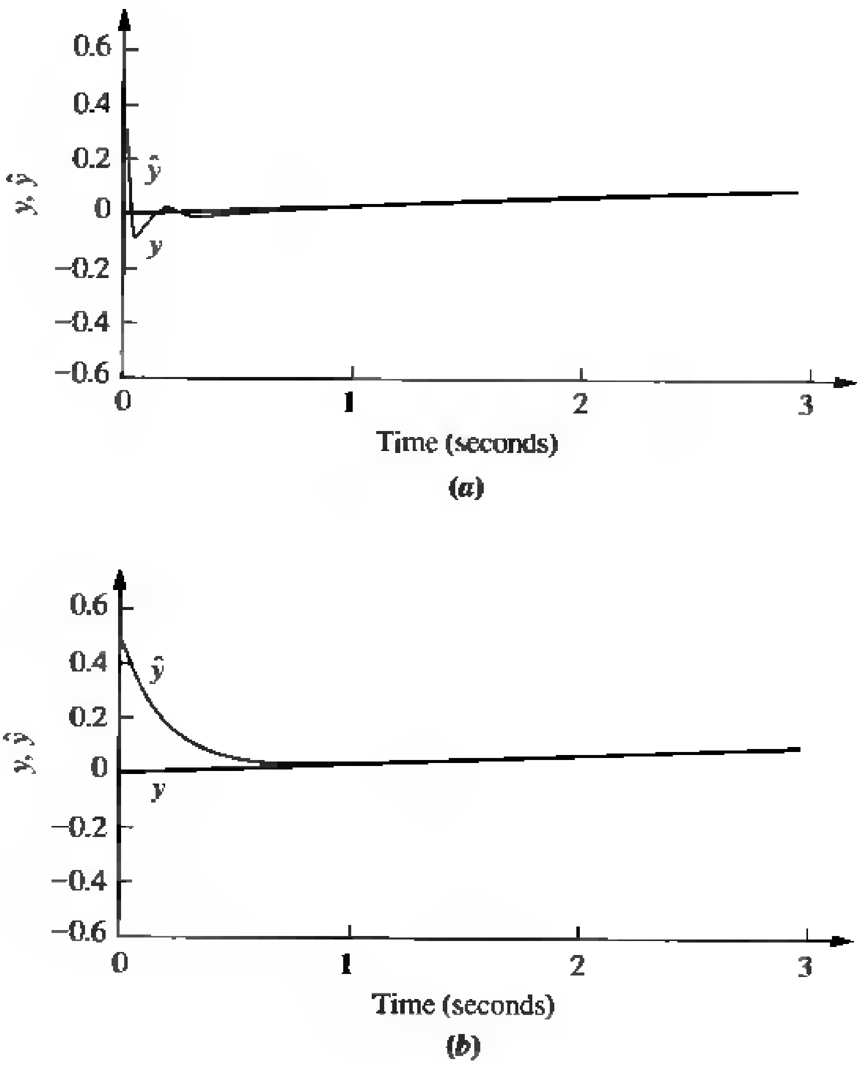


Figure 12.19
Observer design step response simulation:
a. closed-loop observer;
b. open-loop observer with observer gains disconnected



Example 12.9

Observer design by matching coefficients

Problem A time-scaled model for the body's blood glucose level is shown in Eq. (12.105). The output is the deviation in glucose concentration from its mean value in mg/100 ml, and the input is the intravenous glucose injection rate in g/kg/hr (Milhorn, 1966).

$$G(s) = \frac{407(s + 0.916)}{(s + 1.27)(s + 2.69)} \tag{12.105}$$

Design an observer for the phase variables with a transient response described by $\zeta = 0.7$ and $\omega_n = 100$.

Solution We can first model the plant in phase-variable form. The result is shown in Figure 12.20(*a*).

For the plant,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3.42 & -3.96 \end{bmatrix}; \qquad \mathbf{C} = [372.81 \quad 407] \tag{12.106}$$

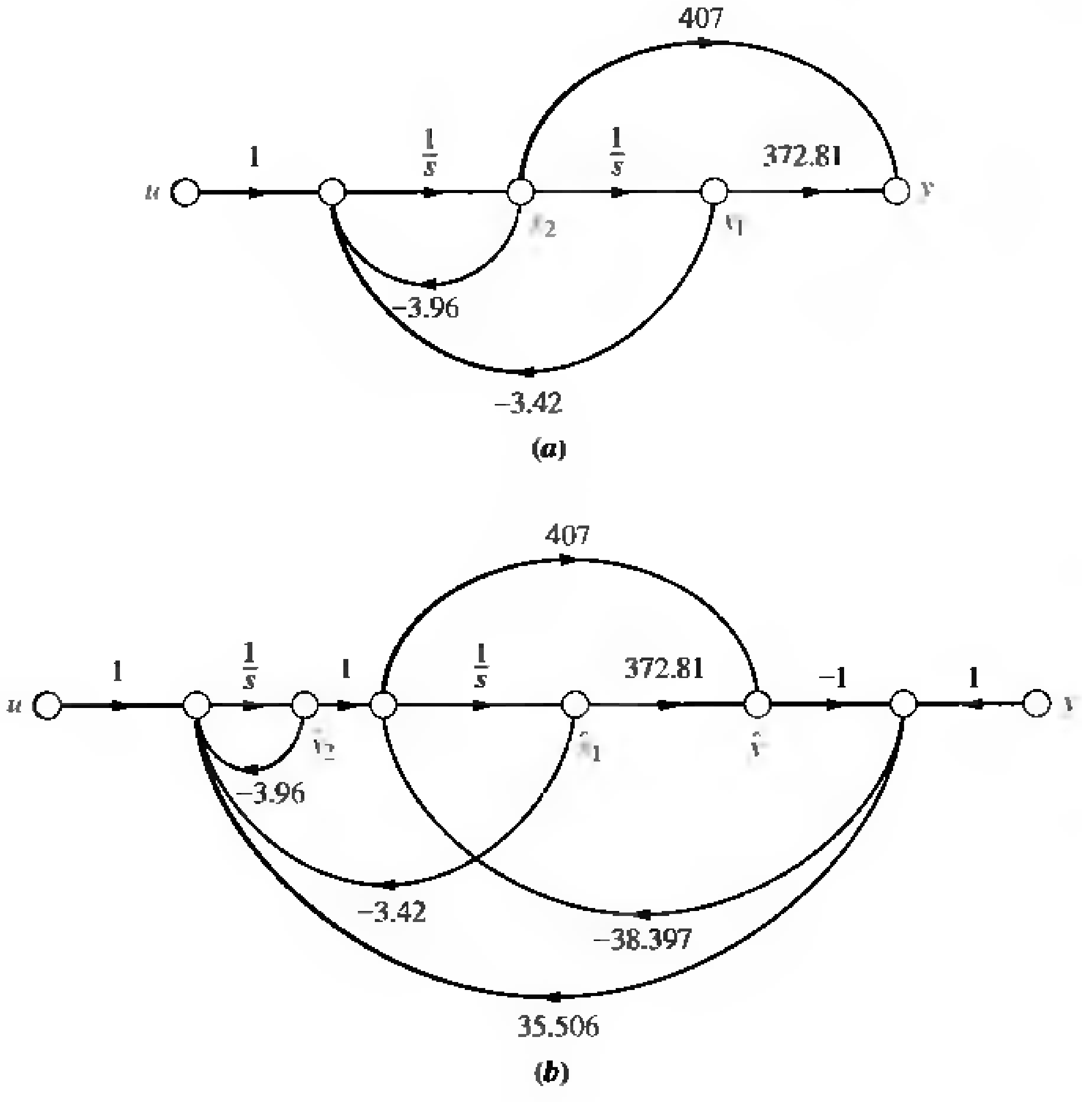


Figure 12.20
a. Plant;
b. designed observer for Example 12.9

Calculation of the observability matrix, $\mathbf{O_M} = [\mathbf{C} \quad \mathbf{CA}]^T$, shows that the plant is observable and we can proceed with the design. Next find the characteristic equation of the observer. First we have

$$\mathbf{A} - \mathbf{LC} = \begin{bmatrix} 0 & 1 \\ -3.42 & -3.96 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [372.81 \quad 407]$$

$$= \begin{bmatrix} -372.81 l_1 & (1 - 407 l_1) \\ -(3.42 + 372.81 l_2) & -(3.96 + 407 l_2) \end{bmatrix} \tag{12.107}$$

Now evaluate $\det[\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})] = 0$ in order to obtain the characteristic equation:

$$\det[\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})] = \det\begin{bmatrix} (\lambda + 372.81 l_1) & -(1 - 407 l_1) \\ (3.42 + 372.81 l_2) & (\lambda + 3.96 + 407 l_2) \end{bmatrix}$$

$$= \lambda^2 + (3.96 + 372.81 l_1 + 407 l_2)\lambda + (3.42 + 84.39 l_1 + 372.81 l_2)$$

$$= 0 \tag{12.108}$$

From the problem statement, we want $\zeta = 0.7$ and $\omega_n = 100$. Thus,

$$\lambda^2 + 140\lambda + 10{,}000 = 0 \tag{12.109}$$

Comparing the coefficients of Eqs. (12.108) and (12.109), we find the values of l_1 and l_2 to be -38.397 and 35.506, respectively. Using Eq. (12.60), where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3.42 & -3.96 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{C} = [372.81 \quad 407]; \quad \mathbf{L} = \begin{bmatrix} -38.397 \\ 35.506 \end{bmatrix} \tag{12.110}$$

the observer is implemented and shown in Figure 12.20(*b*).

Skill-Assessment Exercise 12.6

Control Solution

Problem Design an observer for the plant

$$G(s) = \frac{1}{(s+7)(s+8)(s+9)}$$

whose estimated plant is represented in state space in cascade form as

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}\hat{\mathbf{z}} + \mathbf{B}u = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & -9 \end{bmatrix} \hat{\mathbf{z}} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\hat{y} = \mathbf{C}\hat{\mathbf{x}} = [1 \quad 0 \quad 0]\hat{\mathbf{z}}$$

The closed-loop step response of the observer is to have 10% overshoot with a 0.1 second settling time.

Answer

$$\mathbf{L_z} = \begin{bmatrix} 456 \\ 28{,}640 \\ 1.54 \times 10^6 \end{bmatrix}$$

The complete solution is on the accompanying CD-ROM.

Now that we have explored transient response design using state-space techniques, let us turn to the design of steady-state error characteristics.

12.8 Steady-State Error Design via Integral Control

In Section 7.8 we discussed how to *analyze* systems represented in state space for steady-state error. In this section we discuss how to *design* systems represented in state space for steady-state error.

Consider Figure 12.21. The previously designed controller discussed in Section 12.2 is shown inside the dashed box. A feedback path from the output has been added to form the error, e, which is fed forward to the controlled plant via an integrator. The integrator increases the system type and reduces the previous finite error to zero. We will now derive the form of the state equations for the system of Figure 12.21 and then use that form to design a controller. Thus, we will be able to design a system for zero steady-state error for a step input as well as design the desired transient response.

An additional state variable, x_N, has been added at the output of the leftmost integrator. The error is the derivative of this variable. Now, from Figure 12.21,

$$\dot{x}_N = r - \mathbf{Cx} \tag{12.111}$$

Writing the state equations from Figure 12.21, we have

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u \tag{12.112a}$$

$$\dot{x}_N = -\mathbf{Cx} + r \tag{12.112b}$$

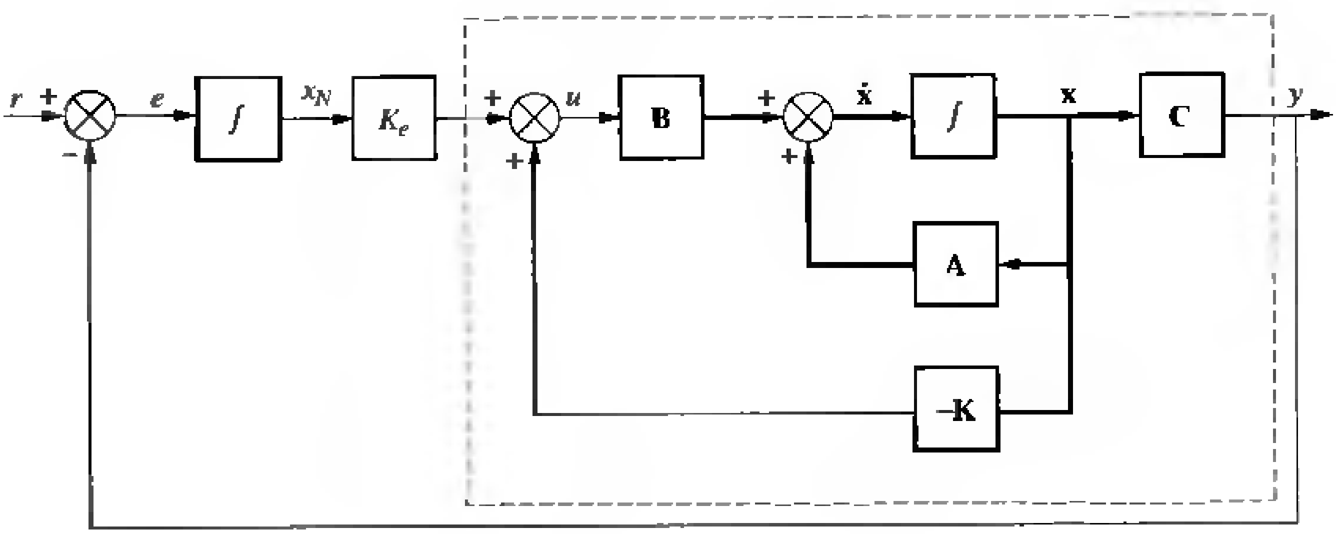
$$y = \mathbf{Cx} \tag{12.112c}$$

Eqs. (12.112) can be written as augmented vectors and matrices. Hence,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \tag{12.113a}$$

$$y = [\mathbf{C} \quad 0] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} \tag{12.113b}$$

Figure 12.21 Integral control for steady-state error design



But

$$u = -\mathbf{K}\mathbf{x} + K_e x_N = -[\mathbf{K} \quad -K_e]\begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} \tag{12.114}$$

Substituting Eq. (12.114) into (12.113*a*) and simplifying, we obtain

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{K}) & \mathbf{B}K_e \\ -\mathbf{C} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r \tag{12.115a}$$

$$y = [\mathbf{C} \quad 0]\begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} \tag{12.115b}$$

Thus, the system type has been increased, and we can use the characteristic equation associated with Eq. (12.115*a*) to design $\mathbf{K}$ and K_e to yield the desired transient response. Realize, we now have an additional pole to place. The effect on the transient response of any closed-loop zeros in the final design must also be taken into consideration. One possible assumption is that the closed-loop zeros will be the same as those of the open-loop plant. This assumption, which of course must be checked, suggests placing higher-order poles at the closed-loop zero locations. Let us demonstrate with an example.

Example 12.10

Design of integral control

Problem Consider the plant of Eqs. (12.116):

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{12.116a}$$

$$y = [1 \quad 0]\mathbf{x} \tag{12.116b}$$

a. Design a controller without integral control to yield a 10% overshoot and a settling time of 0.5 second. Evaluate the steady-state error for a unit step input.

b. Repeat the design of (a) using integral control. Evaluate the steady-state error for a unit step input.

Solution

a. Using the requirements for settling time and percent overshoot, we find that the desired characteristic polynomial is

$$s^2 + 16s + 183.1 \tag{12.117}$$

Since the plant is represented in phase-variable form, the characteristic polynomial for the controlled plant with state-variable feedback is

$$s^2 + (5 + k_2)s + (3 + k_1) \tag{12.118}$$

Equating the coefficients of Eqs. (12.117) and (12.118), we have

$$\mathbf{K} = [k_1 \quad k_2] = [180.1 \quad 11] \tag{12.119}$$

From Eq. (12.3), the controlled plant with state-variable feedback represented in phase-variable form is

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}r = \begin{bmatrix} 0 & 1 \\ -183.1 & -16 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \qquad (12.120a)$$

$$y = \mathbf{Cx} = [1 \quad 0]\mathbf{x} \qquad (12.120b)$$

Using Eq. (7.96), we find that the steady-state error for a step input is

$$\begin{aligned} e(\infty) &= 1 + \mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B} \\ &= 1 + [1 \quad 0]\begin{bmatrix} 0 & 1 \\ -183.1 & -16 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 0.995 \end{aligned} \qquad (12.121)$$

b. We now use Eqs. (12.115) to represent the integral-controlled plant as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ -3 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \quad k_2]\right) & \begin{bmatrix} 0 \\ 1 \end{bmatrix} K_e \\ -[1 \quad 0] & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -(3 + k_1) & -(5 + k_2) & K_e \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad (12.122a)$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} \qquad (12.122b)$$

Using Eq. (3.73) and the plant of Eqs. (12.116), we find that the transfer function of the plant is $G(s) = 1/(s^2 + 5s + 3)$. The desired characteristic polynomial for the closed-loop integral-controlled system is shown in Eq. (12.117). Since the plant has no zeros, we assume no zeros for the closed-loop system and augment Eq. (12.117) with a third pole, $(s + 100)$, which has a real part greater than five times that of the desired dominant second-order poles. The desired third-order closed-loop system characteristic polynomial is

$$(s + 100)(s^2 + 16s + 183.1) = s^3 + 116s^2 + 1783.1s + 18{,}310 \qquad (12.123)$$

The characteristic polynomial for the system of Eqs. (12.122) is

$$s^3 + (5 + k_2)s^2 + (3 + k_1)s + K_e \qquad (12.124)$$

Matching coefficients from Eqs. (12.123) and (12.124), we obtain

$$k_1 = 1780.1 \qquad (12.125a)$$

$$k_2 = 111 \qquad (12.125b)$$

$$K_e = 18{,}310 \qquad (12.125c)$$

Substituting these values into Eqs. (12.122) yields this closed-loop integral-controlled system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_N\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1783.1 & -116 & 18{,}310\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r \qquad (12.126a)$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_N\end{bmatrix} \qquad (12.126b)$$

In order to check our assumption for the zero, we now apply Eq. (3.73) to Eqs. (12.126) and find the closed-loop transfer function to be

$$T(s) = \frac{18{,}310}{s^3 + 116s^2 + 1783.1s + 18{,}310} \qquad (12.127)$$

Since the transfer function matches our design, we have the desired transient response.

Now let us find the steady-state error for a unit step input. Applying Eq. (7.96) to Eqs. (12.126), we obtain

$$e(\infty) = 1 + [1 \quad 0 \quad 0]\begin{bmatrix}0 & 1 & 0\\ -1783.1 & -116 & 18{,}310\\ -1 & 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} = 0 \qquad (12.128)$$

Thus, the system behaves like a Type 1 system.

Skill-Assessment Exercise 12.7

Problem Design an integral controller for the plant

$$\dot{\mathbf{x}} = \begin{bmatrix}0 & 1\\ -7 & -9\end{bmatrix}\mathbf{x} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = [4 \quad 1]\mathbf{x}$$

to yield a step response with 10% overshoot, a peak time of 2 seconds, and zero steady-state error.

Answer $\mathbf{K} = [2.21 \quad -2.7]$, $K_e = 3.79$.

The complete solution is on the accompanying CD-ROM.

Now that we have designed controllers and observers for transient response and steady-state error, we summarize the chapter with a case study demonstrating the design process.

Case Study

Design

Antenna Control: Design of Controller and Observer

In this case study we use our ongoing antenna azimuth position control system to demonstrate the combined design of a controller and an observer. We will assume that the states are not available and must be estimated from the output. The block diagram of the original system is shown on the front endpapers, Configuration 1. Arbitrarily setting the preamplifier gain to 200 and removing the existing feedback, the forward transfer function is simplified to that shown in Figure 12.22.

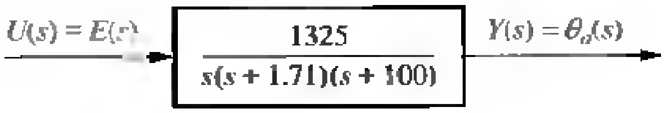


Figure 12.22 Simplified block diagram of antenna control system shown on the front endpapers (Configuration 1) with $K = 200$

The case study will specify a transient response for the system and a faster transient response for the observer. The final design configuration will consist of the plant, the observer, and the controller, as shown conceptually in Figure 12.23. The design of the observer and the controller will be separate.

Problem Using the simplified block diagram of the plant for the antenna azimuth position control system shown in Figure 12.22, design a controller to yield a 10% overshoot and a settling time of 1 second. Place the third pole 10 times as far from the imaginary axis as the second-order dominant pair.

Assume that the state variables of the plant are not accessible and design an observer to estimate the states. The desired transient response for the observer is a 10% overshoot and a natural frequency 10 times as great as the system response above. As in the case of the controller, place the third pole 10 times as far from the imaginary axis as the observer's dominant second-order pair.

Solution

Controller design We first design the controller by finding the desired characteristic equation. A 10% overshoot and a settling time of 1 second yield $\zeta = 0.591$ and $\omega_n = 6.77$. Thus, the characteristic equation for the dominant poles is $s^2 + 8s + 45.8 = 0$, where the dominant poles are located at $-4 \pm j5.46$. The third pole will be 10 times as far from the imaginary axis, or at -40. Hence, the desired characteristic equation for the closed-loop system is

$$(s^2 + 8s + 45.8)(s + 40) = s^3 + 48s^2 + 365.8s + 1832 = 0 \qquad (12.129)$$

Next we find the actual characteristic equation of the closed-loop system. The first step is to model the closed-loop system in state space and then find its characteristic equation. From Figure 12.22, the transfer function of the

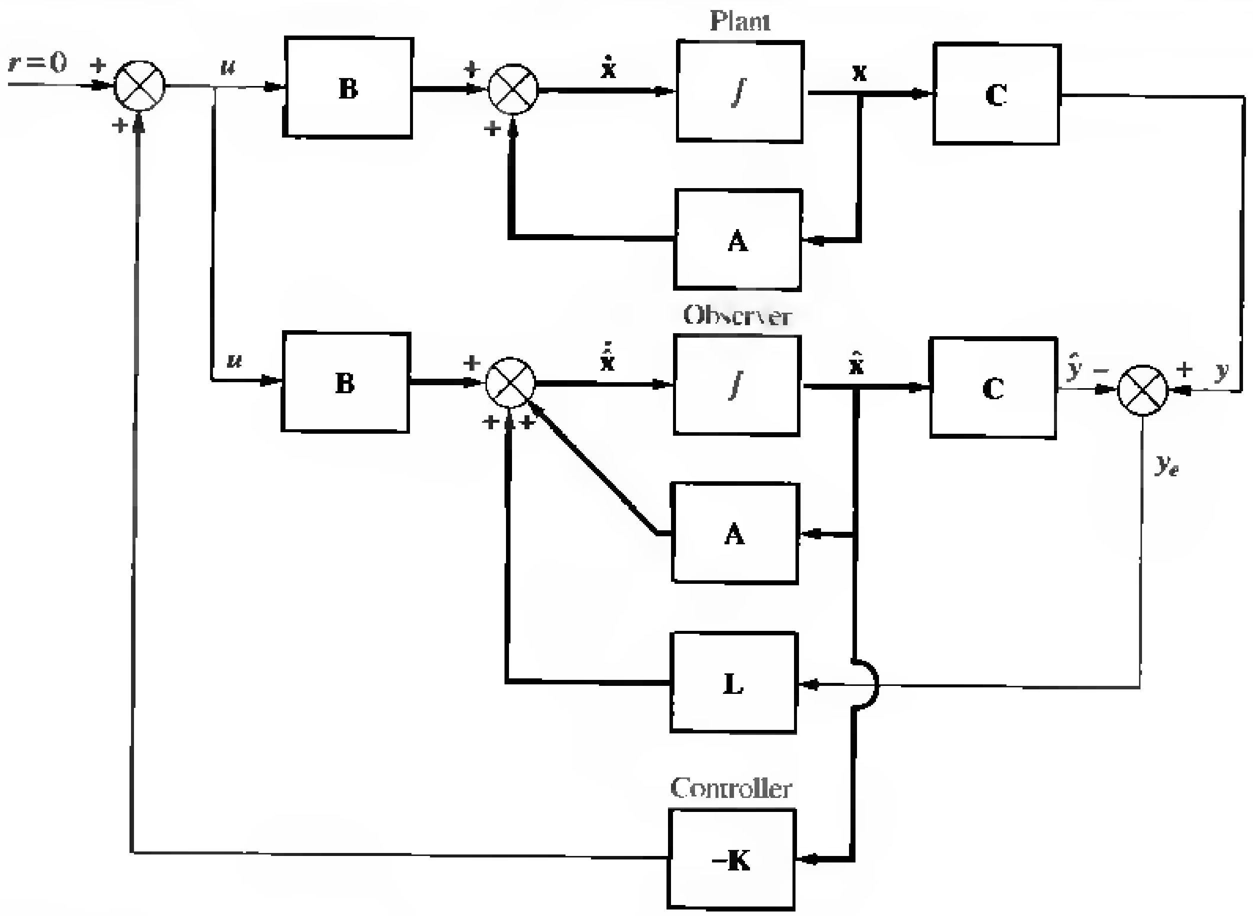


Figure 12.23 Conceptual state-space design configuration, showing plant, observer, and controller

plant is

$$G(s) = \frac{1325}{s(s+1.71)(s+100)} = \frac{1325}{s(s^2+101.71s+171)} \quad (12.130)$$

Using phase variables, this transfer function is converted to the signal-flow graph shown in Figure 12.24, and the state equations are written as follows:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -171 & -101.71 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u = \mathbf{Ax} + \mathbf{B}u \quad (12.131a)$$

$$y = [1325 \quad 0 \quad 0]\mathbf{x} = \mathbf{Cx} \quad (12.131b)$$

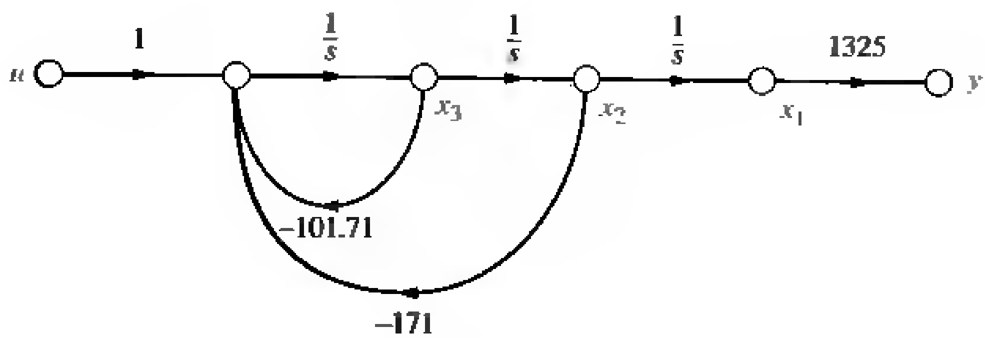


Figure 12.24 Signal-flow graph for $G(s) = 1325/[s(s^2 + 101.71s + 171)]$

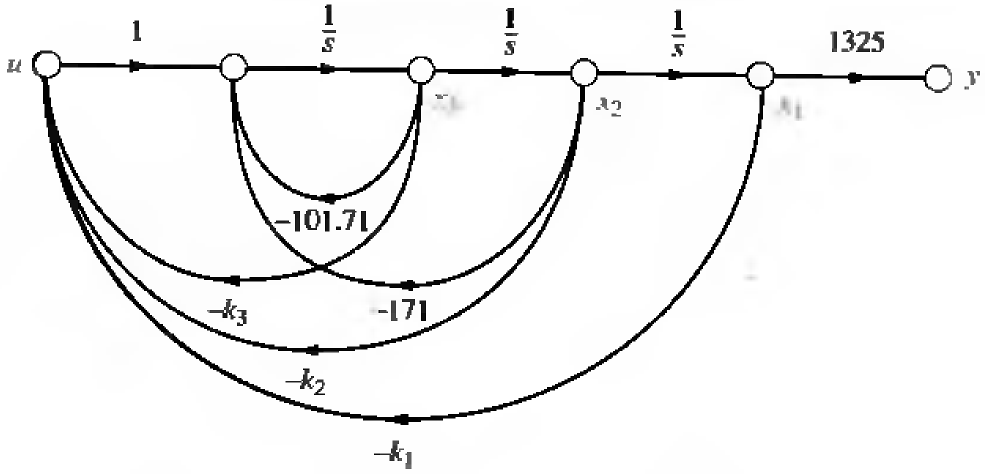


Figure 12.25 Plant with state-variable feedback for controller design

We now pause in our design to evaluate the controllability of the system. The controllability matrix, $\mathbf{C_M}$, is

$$\mathbf{C_M} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -101.71 \\ 1 & -101.71 & 10{,}173.92 \end{bmatrix} \tag{12.132}$$

The determinant of $\mathbf{C_M}$ is -1; thus, the system is controllable.

Continuing with the design of the controller, we show the controller's configuration with the feedback from all state variables in Figure 12.25. We now find the characteristic equation of the system of Figure 12.25. From Eq. (12.7) and Eq. (12.131*a*), the system matrix, $\mathbf{A} - \mathbf{BK}$, is

$$\mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -(171 + k_2) & -(101.71 + k_3) \end{bmatrix} \tag{12.133}$$

Thus, the closed-loop system's characteristic equation is

$$\det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})] = s^3 + (101.71 + k_3)s^2 + (171 + k_2)s + k_1 = 0 \tag{12.134}$$

Matching the coefficients of Eq. (12.129) with those of Eq. (12.134), we evaluate the k_i's as follows:

$$k_1 = 1832 \tag{12.135a}$$

$$k_2 = 194.8 \tag{12.135b}$$

$$k_3 = -53.71 \tag{12.135c}$$

Observer design Before designing the observer, we test the system for observability. Using the **A** and **C** matrices from Eqs. (12.131), the observability matrix, $\mathbf{O_M}$, is

$$\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1325 & 0 & 0 \\ 0 & 1325 & 0 \\ 0 & 0 & 1325 \end{bmatrix} \tag{12.136}$$

The determinant of $\mathbf{O_M}$ is 1325^3. Thus, $\mathbf{O_M}$ is of rank 3, and the system is observable.

We now proceed to design the observer. Since the order of the system is not high, we will design the observer directly without first converting to observer canonical form. From Eq. (12.64*a*) we need first to find $\mathbf{A} - \mathbf{LC}$. $\mathbf{A}$ and $\mathbf{C}$ from Eqs. (12.131) along with

$$\mathbf{L} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \tag{12.137}$$

are used to evaluate $\mathbf{A} - \mathbf{LC}$ as follows:

$$\mathbf{A} - \mathbf{LC} = \begin{bmatrix} -1325l_1 & 1 & 0 \\ -1325l_2 & 0 & 1 \\ -1325l_3 & -171 & -101.71 \end{bmatrix} \tag{12.138}$$

The characteristic equation for the observer is now evaluated as

$$\begin{aligned} \det[\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})] &= \lambda^3 + (1325l_1 + 101.71)\lambda^2 \\ &\quad + (134{,}800l_1 + 1325l_2 + 171)\lambda \\ &\quad + (226{,}600l_1 + 134{,}800l_2 + 1325l_3) \\ &= 0 \end{aligned} \tag{12.139}$$

From the problem statement, the poles of the observer are to be placed to yield a 10% overshoot and a natural frequency 10 times that of the system's dominant pair of poles. Thus, the observer's dominant poles yield $[s^2 + (2 \times 0.591 \times 67.7)s + 67.7^2] = (s^2 + 80s + 4583)$. The real part of the roots of this polynomial is -40. The third pole is then placed 10 times farther from the imaginary axis at -400. The composite characteristic equation for the observer is

$$\begin{aligned} (s^2 + 80s + 4583)(s + 400) &= s^3 + 480s^2 + 36.580s \\ &\quad + 1{,}833{,}000 = 0 \end{aligned} \tag{12.140}$$

Matching coefficients from Eqs. (12.139) and (12.140), we solve for the observer gains:

$$l_1 = 0.286 \tag{12.141a}$$

$$l_2 = -1.57 \tag{12.141b}$$

$$l_3 = 1494 \tag{12.141c}$$

Figure 12.26, which follows the general configuration of Figure 12.23, shows the completed design, including the controller and the observer.

The results of the design are shown in Figure 12.27. Figure 12.27(*a*) shows the impulse response of the closed-loop system without any difference between

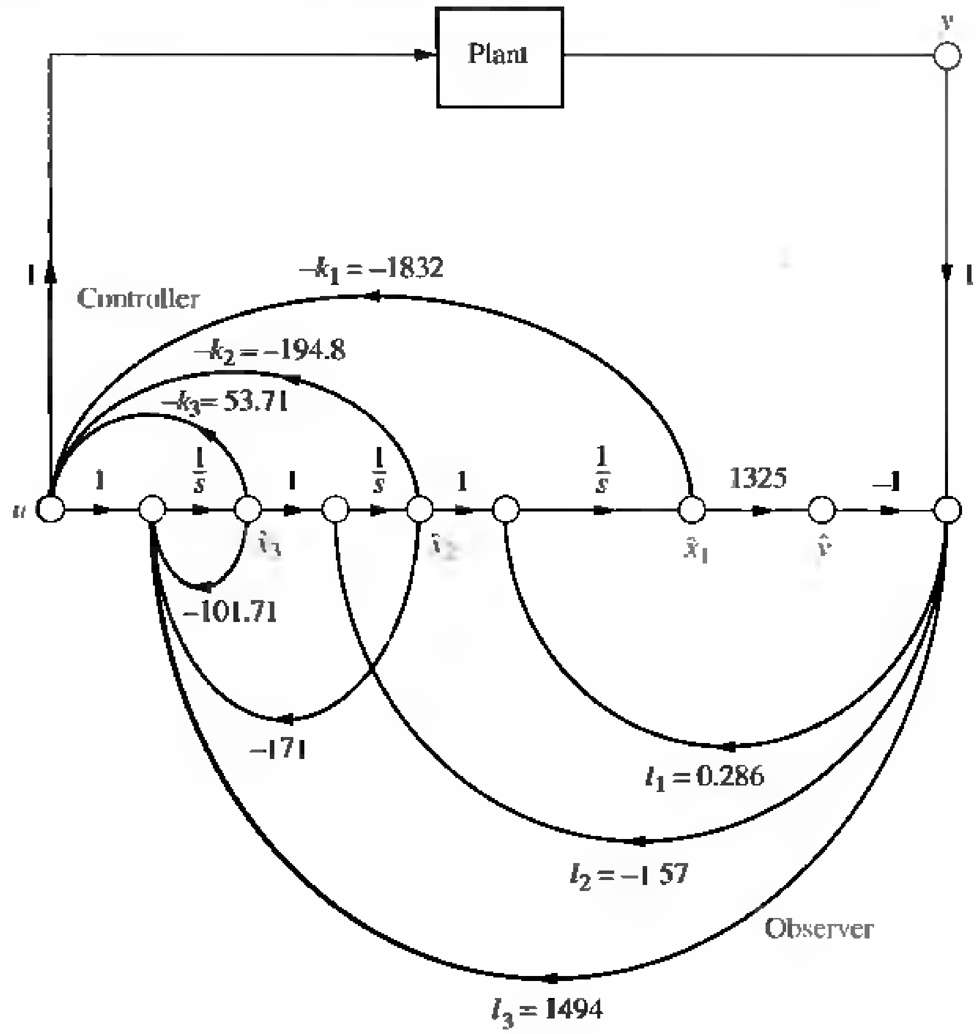


Figure 12.26 Completed state-space design for the antenna azimuth position control system, showing controller and observer

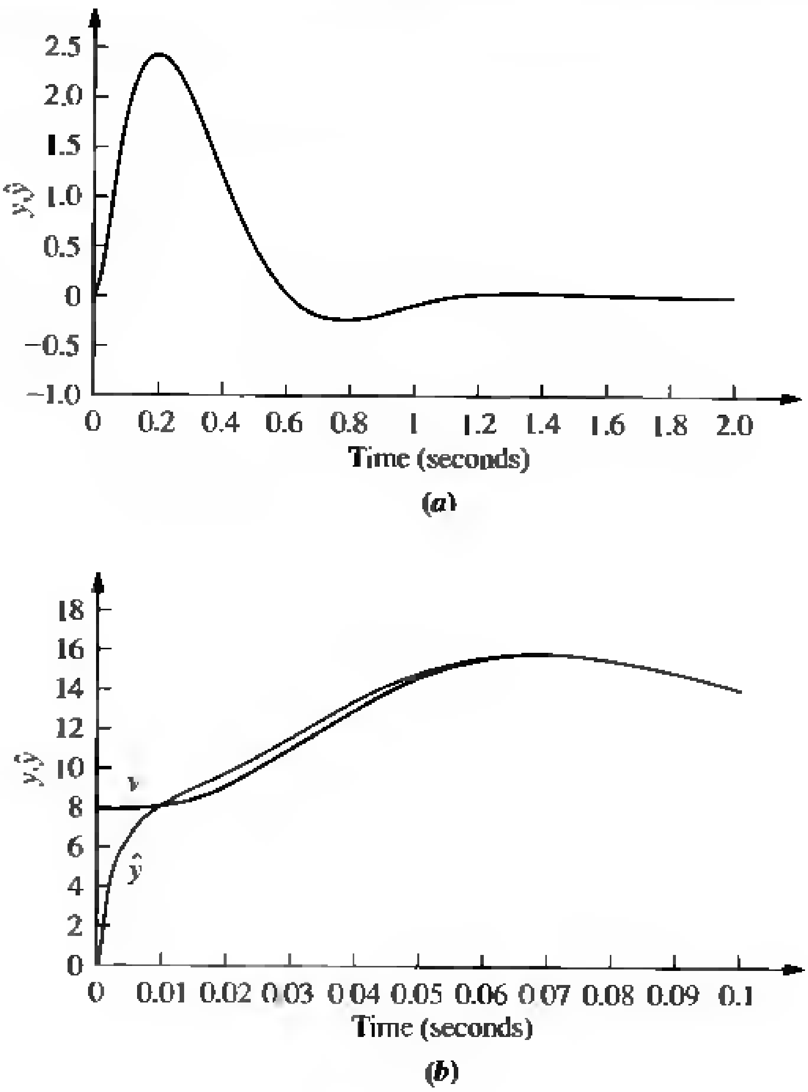
the plant and its modeling as an observer. The undershoot and settling time approximately meet the requirements set forth in the problem statement of 10% and 1 second, respectively. In Figure 12.27(*b*), we see the response designed into the observer. An initial condition of 0.006 was given to x_1 in the plant to make the modeling of the plant and observer different. Notice that the observer's response follows the plant's response by the time 0.06 second is reached.

Challenge You are now given a case study to test your knowledge of this chapter's objectives: You are given the antenna azimuth position control system shown on the front endpapers, Configuration 3. If the preamplifier gain $K = 20$, do the following:

a. Design a controller to yield 15% overshoot and a settling time of 2 seconds. Place the third pole 10 times as far from the imaginary axis as the second-order dominant pole pair. Use physical variables as follows: power amplifier output, motor angular velocity, and motor displacement.

b. Redraw the schematic shown on the front endpapers, showing a tachometer that yields rate feedback along with any added gains or attenuators required to implement the state-variable feedback gains.

Figure 12.27 Designed response of antenna azimuth position control system: **a.** impulse response—plant and observer with the same initial conditions, $x_1(0) = \hat{x}_1(0) = 0$; **b.** portion of impulse response—plant and observer with different initial conditions, $x_1(0) = 0.006$ for the plant, $\hat{x}_1(0) = 0$ for the observer



c. Assume that the tachometer is not available to provide rate feedback. Design an observer to estimate the physical variables' states. The observer will respond with 10% overshoot and a natural frequency 10 times as great as the system response. Place the observer's third pole 10 times as far from the imaginary axis as the observer's dominant second-order pole pair.

d. Redraw the schematic on the front endpapers, showing the implementation of the controller and the observer.

MATLAB

e. Repeat (a) and (c) using MATLAB.

Summary

This chapter has followed the path established by Chapters 9 and 11—control system design. Chapter 9 used root locus techniques to design a control system with a desired transient response. Sinusoidal frequency response techniques for design were covered in Chapter 11, and in this chapter we used state-space design techniques.

State-space design consists of specifying the system's desired pole locations and then designing a controller consisting of state-variable feedback gains to meet these requirements. If the state variables are not available, an observer is designed to emulate the plant and provide estimated state variables.

Controller design consists of feeding back the state variables to the input, u, of the system through specified gains. The values of these gains are found by matching the coefficients of the system's characteristic equation with the coefficients of the desired characteristic equation. In some cases the control signal, u, cannot affect one or more state variables. We call such a system *uncontrollable*. For this system a total design is not possible. Using the controllability matrix, a designer can tell whether or not a system is controllable prior to the design.

Observer design consists of feeding back the error between the actual output and the estimated output. This error is fed back through specified gains to the derivatives of the estimated state variables. The values of these gains are also found by matching the coefficients of the observer's characteristic equation with the coefficients of the desired characteristic equation. The response of the observer is designed to be faster than that of the controller, so the estimated state variables effectively appear instantaneously at the controller. For some systems the state variables cannot be deduced from the output of the system, as is required by the observer. We call such systems *unobservable*. Using the observability matrix, the designer can tell whether or not a system is observable. Observers can be designed only for observable systems.

Finally, we discussed ways of improving the steady-state error performance of systems represented in state space. The addition of an integration before the controlled plant yields improvement in the steady-state error. In this chapter this additional integration was incorporated into the controller design.

Three advantages of state-space design are apparent. First, in contrast to the root locus method, all pole locations can be specified to ensure a negligible effect of the nondominant poles upon the transient response. With the root locus, we were forced to justify an assumption that the nondominant poles did not appreciably affect the transient response. We were not always able to do so. Second, with the use of an observer, we are no longer forced to acquire the actual system variables for feedback. The advantage here is that sometimes the variables cannot be physically accessed, or it may be too expensive to provide that access. Finally, the methods shown lend themselves to design automation using the digital computer.

A disadvantage of the design methods covered in this chapter is the designer's inability to design the location of open- or closed-loop zeros that may affect the transient response. In root locus or frequency response design, the zeros of the lag or lead compensator can be specified. Another disadvantage of state-space methods concerns the designer's ability to relate all pole locations to the desired response; this relationship is not always apparent. Also, once the design is completed, we may not be satisfied with the sensitivity to parameter changes.

Finally, as previously discussed, state-space techniques do not satisfy our intuition as much as root locus techniques, where the effect of parameter changes can be immediately seen as changes in closed-loop pole locations.

In the next chapter we return to the frequency domain and design digital systems using gain adjustment and cascade compensation.

Review Questions

1. Briefly describe an advantage that state-space techniques have over root locus techniques in the placement of closed-loop poles for transient response design.
2. Briefly describe the design procedure for a controller.
3. Different signal-flow graphs can represent the same system. Which form facilitates the calculation of the variable gains during controller design?
4. In order to effect a complete controller design, a system must be controllable. Describe the physical meaning of controllability.
5. Under what conditions can inspection of the signal-flow graph of a system yield immediate determination of controllability?
6. In order to determine controllability mathematically, the controllability matrix is formed, and its rank evaluated. What is the final step in determining controllability if the controllability matrix is a square matrix?
7. What is an observer?
8. Under what conditions would you use an observer in your state-space design of a control system?
9. Briefly describe the configuration of an observer.
10. What plant representation lends itself to easier design of an observer?
11. Briefly describe the design technique for an observer, given the configuration you described in Question 9.
12. Compare the major difference in the transient response of an observer to that of a controller. Why does this difference exist?
13. From what equation do we find the characteristic equation of the controller-compensated system?
14. From what equation do we find the characteristic equation of the observer?
15. In order to effect a complete observer design, a system must be observable. Describe the physical meaning of observability.
16. Under what conditions can inspection of the signal-flow graph of a system yield immediate determination of observability?
17. In order to determine observability mathematically, the observability matrix is formed and its rank evaluated. What is the final step in determining observability if the observability matrix is a square matrix?

Problems

1. Consider the following open-loop transfer functions, where $G(s) = Y(s)/U(s)$, $Y(s)$ is the Laplace transform of the output, and $U(s)$ is the Laplace transform of the input control signal:

   i. $G(s) = \dfrac{(s+3)}{(s+4)^2}$

   ii. $G(s) = \dfrac{s}{(s+5)(s+7)}$

   iii. $G(s) = \dfrac{20s(s+7)}{(s+3)(s+7)(s+9)}$

   iv. $G(s) = \dfrac{30(s+2)(s+3)}{(s+4)(s+5)(s+6)}$

   v. $G(s) = \dfrac{s^2+8s+15}{(s^2+4s+10)(s^2+3s+12)}$

   For each of these transfer functions, do the following:

   a. Draw the signal-flow graph in phase-variable form.

   b. Add state-variable feedback to the signal-flow graph.

   c. For each closed-loop signal-flow graph, write the state equations.

   d. Write, *by inspection*, the closed-loop transfer function, $T(s)$, for your closed-loop signal-flow graphs.

   e. Verify your answers for $T(s)$ by finding the closed-loop transfer functions from the state equations and Eq. (3.73).

Control Solution

2. The following open-loop transfer functions can be represented by signal-flow graphs in cascade form.

   i. $G(s) = \dfrac{30(s+2)(s+7)}{s(s+3)(s+5)}$

   ii. $G(s) = \dfrac{5(s^2+3s+7)}{(s+2)(s^2+2s+10)}$

   For each, do the following:

   a. Draw the signal-flow graph and show the state-variable feedback.

   b. Find the closed-loop transfer function with state-variable feedback.

3. The following open-loop transfer functions can be represented by signal-flow graphs in parallel form.

   i. $G(s) = \dfrac{50(s^2+7s+25)}{s(s+10)(s+20)}$

   ii. $G(s) = \dfrac{50(s+3)(s+4)}{(s+5)(s+6)(s+7)}$

For each, do the following:

a. Draw the signal-flow graph and show the state-variable feedback.

b. Find the closed-loop transfer function with state-variable feedback.

4. Given the following open-loop plant,

$$G(s) = \frac{20}{(s+1)(s+3)(s+7)}$$

design a controller to yield a 15% overshoot and a settling time of 0.75 second. Place the third pole 10 times as far from the imaginary axis as the dominant pole pair. Use the phase variables for state-variable feedback.

5. Section 12.2 showed that controller design is easier to implement if the uncompensated system is represented in phase-variable form with its typical lower companion matrix. We alluded to the fact that the design can just as easily progress using the controller canonical form with its upper companion matrix.

a. Redo the general controller design covered in Section 12.2, assuming that the plant is represented in controller canonical form rather than phase-variable form.

b. Apply your derivation to Example 12.1 if the uncompensated plant is represented in controller canonical form.

Control Solution

6. Given the following open-loop plant:

$$G(s) = \frac{100(s+2)(s+20)}{(s+1)(s+3)(s+4)}$$

design a controller to yield 15% overshoot with a peak time of 0.5 second. Use the controller canonical form for state-variable feedback.

7. Given the following open-loop plant:

$$G(s) = \frac{20(s+2)}{s(s+4)(s+6)}$$

design a controller to yield a 10% overshoot and a settling time of 2 seconds. Place the third pole 10 times as far from the imaginary axis as the dominant pole pair. Use the phase variables for state-variable feedback.

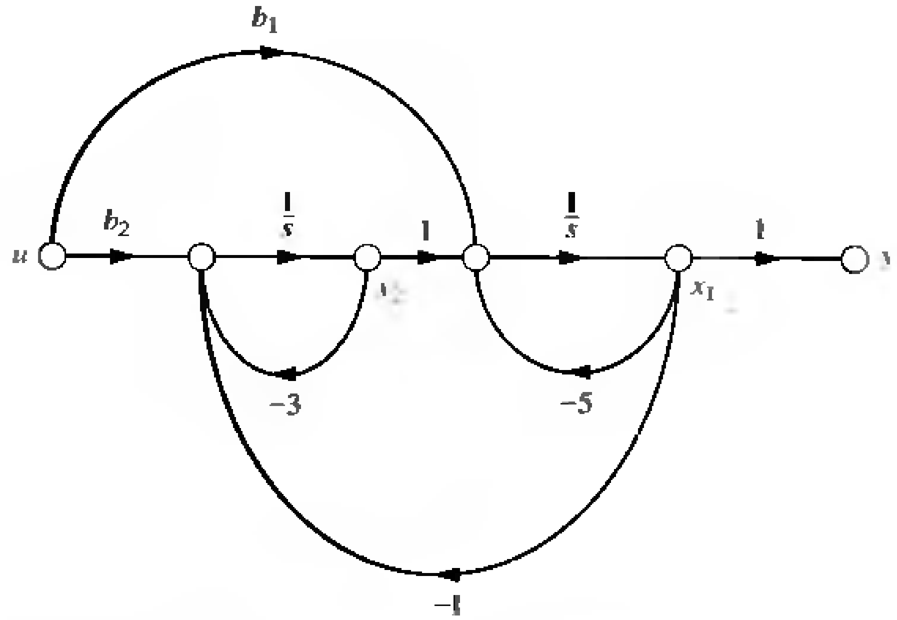
8. Repeat Problem 4 assuming that the plant is represented in the cascade form. Do not convert to phase-variable form.

9. Repeat Problem 7 assuming that the plant is represented in the parallel form. Do not convert to phase-variable form.

Control Solution

10. Given the plant shown in Figure P12.1, what relationship exists between b_1 and b_2 to make the system uncontrollable?

Figure P12.1



11. For each of the plants represented by signal-flow graphs in Figure P12.2, determine the controllability. If the controllability can be determined by inspection, state that it can and then verify your conclusions using the controllability matrix.

Figure P12.2
(figure continues)

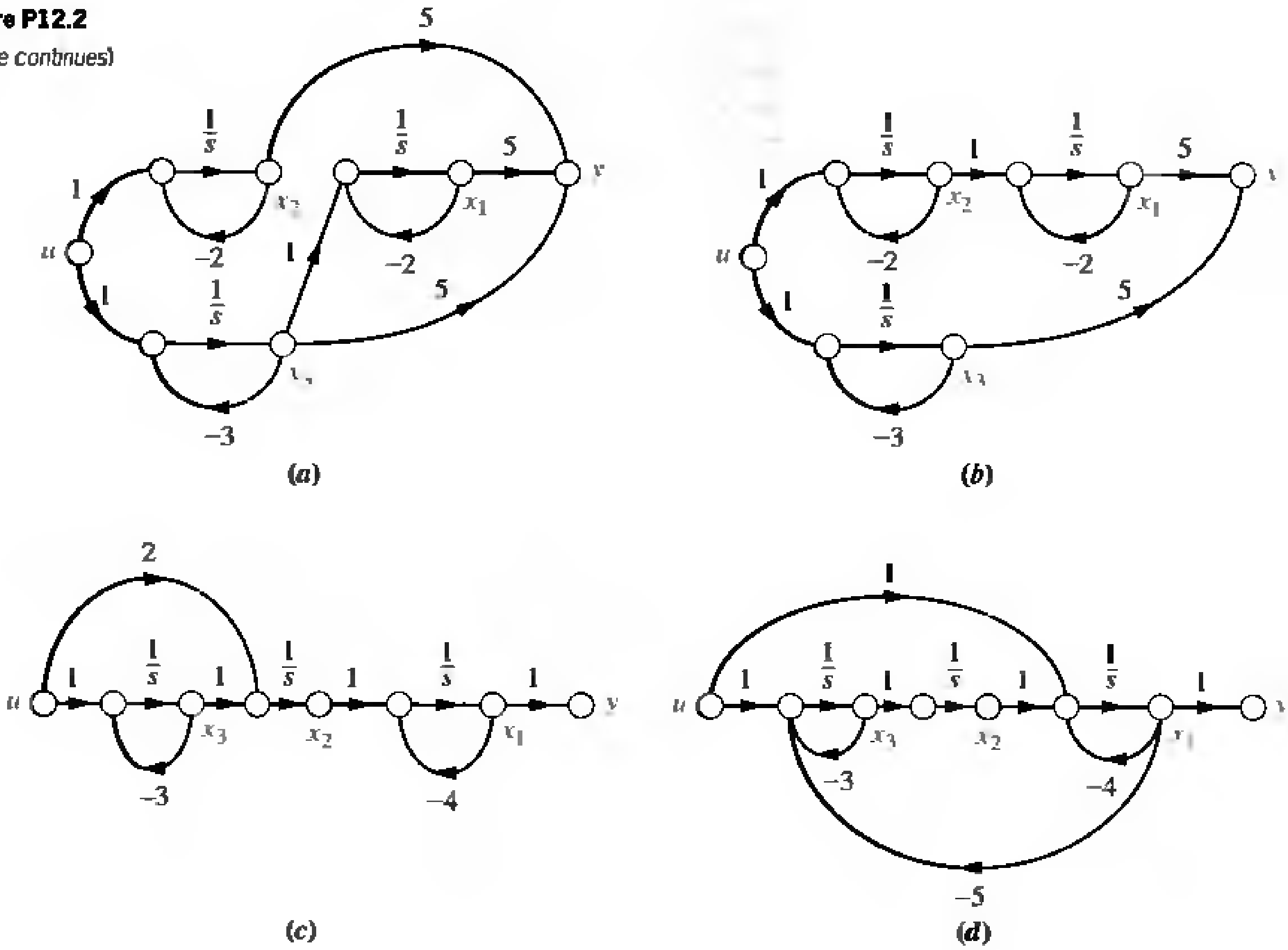
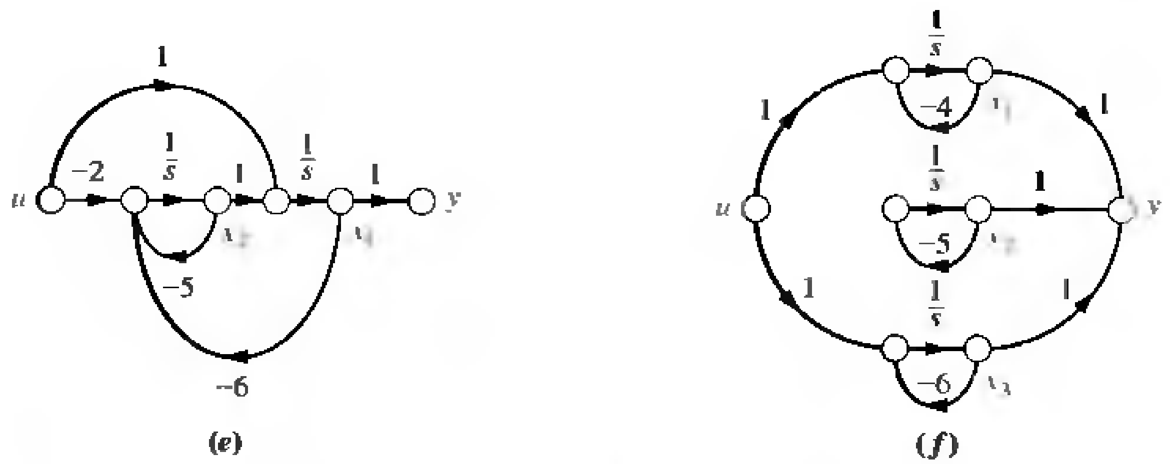


Figure P12.2 (*continued*)



MATLAB

12. Use MATLAB to determine the controllability of the systems of Figure P12.2(d) and (f).

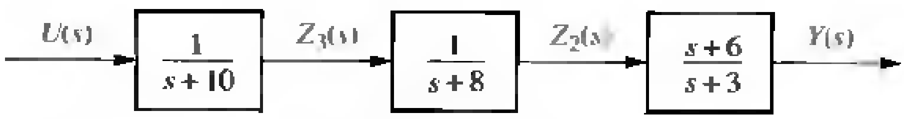
13. In Section 12.4 we discussed how to design a controller for systems not represented in phase-variable form with its typical lower companion matrix. We described how to convert the system to phase-variable form, design the controller, and convert back to the original representation. This technique can be applied just as easily if the original representation is converted to controller canonical form with its typical upper companion matrix. Redo Example 12.4 in the text by designing the controller after converting the uncompensated plant to controller canonical form.

14. Consider the following transfer function:

$$G(s) = \frac{(s+6)}{(s+3)(s+8)(s+10)}$$

If the system is represented in cascade form, as shown in Figure P12.3, design a controller to yield a closed-loop response of 10% overshoot with a settling time of 1 second. Design the controller by first transforming the plant to phase variables.

Figure P12.3



MATLAB

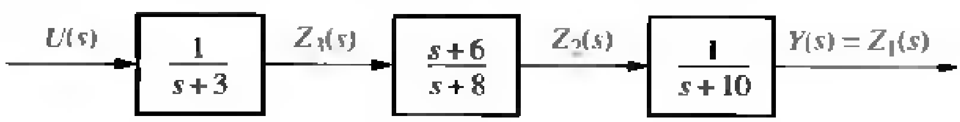
15. Use MATLAB to design the controller gains for the system given in Problem 14.

16. Repeat Problem 14 assuming that the plant is represented in parallel form.

Control Solution

17. The open-loop system of Problem 14 is represented as shown in Figure P12.4. If the output of each block is assigned to be a state variable, design the controller gains for feedback from these state variables.

Figure P12.4



18. If an open-loop plant,

$$G(s) = \frac{100}{s(s+4)(s+8)}$$

is represented in parallel form, design a controller to yield a closed-loop response of 15% overshoot and a peak time of 0.2 second. Design the controller by first transforming the plant to controller canonical form.

19. Consider the plant

$$G(s) = \frac{1}{s(s+3)(s+7)}$$

whose state variables are not available. Design an observer for the observer canonical variables to yield a transient response described by $\zeta = 0.4$ and $\omega_n = 75$. Place the third pole 10 times farther from the imaginary axis than the dominant poles.

20. Design an observer for the plant

$$G(s) = \frac{10}{(s+2)(s+6)(s+12)}$$

operating with 10% overshoot and 2 seconds peak time. Design the observer to respond 10 times as fast as the plant. Place the observer third pole 20 times as far from the imaginary axis as the observer dominant poles. Assume the plant is represented in observer canonical form.

21. Repeat Problem 19 assuming that the plant is represented in phase-variable form. Do not convert to observer canonical form.

Control Solution

22. Consider the plant

$$G(s) = \frac{(s+2)}{(s+5)(s+9)}$$

whose phase variables are not available. Design an observer for the phase variables with a transient response described by $\zeta = 0.6$ and $\omega_n = 120$. Do not convert to observer canonical form.

23. Determine whether or not each of the systems shown in Figure P12.2 is observable.

MATLAB

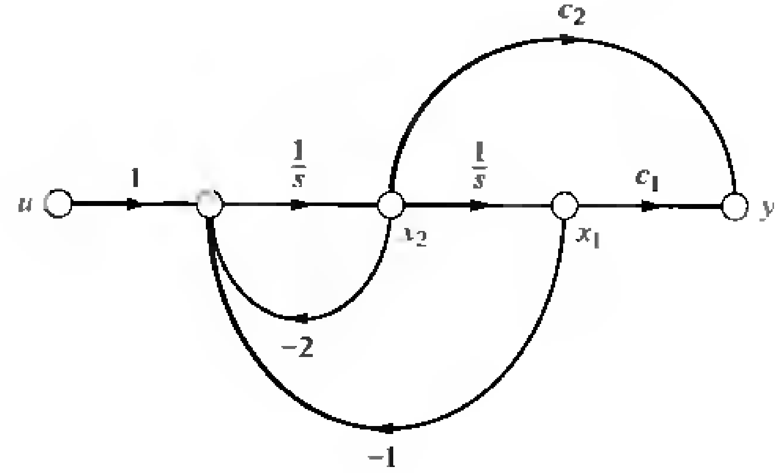
24. Use MATLAB to determine the observability of the systems of Figure P12.2(a) and (f).

25. Given the plant of Figure P12.5, what relationship must exist between c_1 and c_2 in order for the system to be unobservable?

26. Design an observer for the plant

$$G(s) = \frac{1}{(s+5)(s+13)(s+20)}$$

Figure P12.5



represented in cascade form. Transform the plant to observer canonical form for the design. Then transform the design back to cascade form. The characteristic polynomial for the observer is to he $s^3 + 600s^2 + 40{,}000s + 1{,}500{,}000$.

MATLAB

27. Use MATLAB to design the observer gains for the system given in Problem 26.

28. Repeat Problem 26 assuming that the plant is represented in parallel form.

29. Design an observer for

$$G(s) = \frac{50}{(s+3)(s+6)(s+9)}$$

represented in phase-variable form with a desired performance of 10% overshoot and a settling time of 0.5 second. The observer will be 10 times as fast as the plant, and the observer's nondominant pole will be 10 times as far from the imaginary axis as the observer's dominant poles. Design the observer by first converting to observer canonical form.

Control Solution

30. Given the plant

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u; \qquad y = [1 \quad 1]\mathbf{x}$$

design an integral controller to yield a 10% overshoot, 0.5-second settling time, and zero steady-state error for a step input.

31. Repeat Problem 30 for the following plant:

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u; \qquad y = [1 \quad 1]\mathbf{x}$$

Design Problems

32. A magnetic levitation system is described in Problem 42 in Chapter 9 (Cho, 1993). Remove the photocell in Figure P9.11(*b*) and design a controller for phase variables to yield a step response with 5% overshoot and a settling time of 0.5 second.

33. The conceptual block diagram of a gas-fired heater is shown in Figure P12.6. The commanded fuel pressure is proportional to the desired temperature. The difference between the commanded fuel pressure and a measured pressure related to the output temperature is used to actuate a valve and release fuel to the heater. The rate of fuel flow determines the temperature. When the output temperature equals the equivalent commanded temperature as determined by the commanded fuel pressure, the fuel flow is stopped and the heater shuts off (Tyner, 1968).

If the transfer function of the heater, $G_H(s)$, is

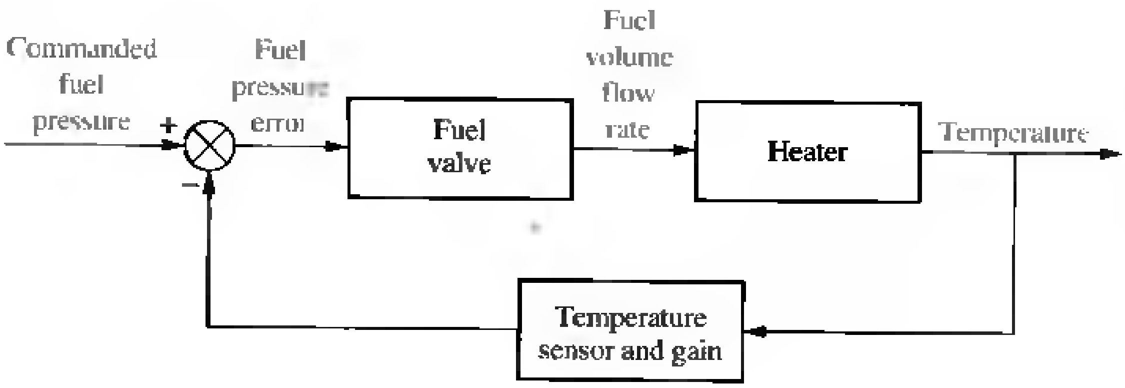
$$G_H(s) = \frac{1}{(s+0.4)(s+0.8)} \frac{\text{degrees F}}{\text{ft}^3/\text{min}}$$

and the transfer function of the fuel valve, $G_v(s)$, is

$$G_v(s) = \frac{5}{s+5} \frac{\text{ft}^3/\text{min}}{\text{psi}}$$

replace the temperature feedback path with a phase-variable controller that yields a 5% overshoot and a settling time of 10 minutes. Also, design an observer that will respond 10 times faster than the system but with the same percent overshoot.

Figure P12.6
Block diagram of a gas-fired heater



Control Solution

34. The floppy disk drive of Problem 46 in Chapter 8 is to be redesigned using state-variable feedback. The controller is replaced by a unity dc gain amplifier, $G_a(s) = 800/(s+800)$. The plant, $G_p(s) = 20{,}000/[s(s+100)]$, is in cascade with the amplifier.

a. Design a controller to yield 10% overshoot and a settling time of 0.05 second. Assume that the state variables are the output position, output velocity, and amplifier output.

b. Evaluate the steady-state error and redesign the system with an integral controller to reduce the steady-state error to zero. (Use of a program with symbolic capability is highly recommended.)

MATLAB

c. Simulate the step response for both the controller-compensated and integral controller-compensated systems. Use MATLAB or any other computer program.

MATLAB

35. Given the angle of attack control system for the AFTI/F-16 aircraft shown in Figure P9.12 (Monahemi, 1992), use MATLAB to design a controller for the plant to yield 10% overshoot with a settling time of 0.5 second. Assume that the phase variables are accessible. Have the program display the step response of the compensated system.

MATLAB

36. For the angle of attack control system of Problem 35, use MATLAB to design an observer for the phase variables that is 15 times faster than the controller designed system.

37. For the angle of attack control system of Problem 35, do the following:

    a. Design an integral control using phase variables to reduce the steady-state error to zero. (Use of a program with symbolic capability is highly recommended.)

MATLAB

    b. Use MATLAB to obtain the step response.

Progressive Analysis and Design Problem

38. **High-speed rail pantograpb.** Problem 17 in Chapter 1 discusses active control of a pantograph mechanism for high-speed rail systems (O'Connor, 1997). In Problem 62(a), Chapter 5, you found the block diagram for the active pantograph control system. For the open-loop portion of the pantograph system modeled in Chapter 5, do the following:

    a. Design a controller to yield 20% overshoot and a 1-second settling time.

    b. Repeat (a) with a zero steady-state error.

Cyber Exploration Laboratory

Experiment 12.1

Objective To simulate a system that has been designed for transient response via a state-space controller and observer.

Minimum Required Software Packages MATLAB, Simulink, and the Control System Toolbox

Prelab:

1. This experiment is based upon your design of a controller and observer as specified in the Case Study Challenge problem in Chapter 12. Once you have completed the controller and observer design in that problem, go on to Prelab 2.

2. What is the controller gain vector for your design of the system specified in the Case Study Challenge problem in Chapter 12?

3. What is the observer gain vector for your design of the system specified in the Case Study Challenge problem in Chapter 12?

4. Draw a Simulink diagram to simulate the system. Show the system, the controller, and the observer using the physical variables specified in the Case Study Challenge problem in Chapter 12.

Lab:

1. Using Simulink and your diagram from Prelab 4, produce the Simulink diagram from which you can simulate the response.
2. Produce response plots of the system and the observer for a step input.
3. Measure the percent overshoot and the settling time for both plots.

Postlab:

1. Make a table showing the design specifications and the simulation results for percent overshoot and settling time.
2. Compare the design specifications with the simulation results for both the system response and the observer response. Explain any discrepancies
3. Describe any problems you had implementing your design.

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